Generalized Trigonometric Functions

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1 Introduction

We give a definition of generalized trigonometric functions. The notations are due to Lang and Edmunds [1]. For real numbers $p, q > 1$, we define a function F_{pq} by

$$F_{pq}(y) = \int_0^y (1-t^q)^{-1/p} dt, \tag{1.1}$$

where $y \in [0,1]$. We introduce generalized pi's as follows:

$$\pi_{pq} = 2F_{pq}(1) = \frac{2}{q} B\Big(\frac{1}{p^*}, \frac{1}{q}\Big) = \frac{p^*}{q} \pi_{q^* p^*}, \tag{1.2}$$

where $1/p^* = 1 - 1/p$ and $B(.,.)$ denotes the beta function. We define a function $y = \sin_{pq} t$ on $[0, \pi_{pq}/2]$ by the inverse function of $t = F_{pq}(y)$, called generalized sine function. We define a function $x = \cos_{pq} t$ on $[0, \pi_{pq}/2]$ by the following equality:

$$\cos_{pq} t = (1 - (\sin_{pq} t)^q)^{1/p}, \tag{1.3}$$

called generalized cosine function. We can define $\sin_{pq} t,\ \cos_{pq} t$ on the whole real numbers by the following equalities:

$$\begin{aligned} &\sin_{pq} t = \sin_{pq}(\pi_{pq} - t),\ \sin_{pq} t = -\sin_{pq}(-t), \\ &\cos_{pq} t = -\cos_{pq}(\pi_{pq} - t),\ \cos_{pq} t = \cos_{pq}(-t). \end{aligned} \tag{1.4}$$

When $p = q$, we abbreviate $\sin_{pp} t$ to $\sin_p t$, $\cos_{pp} t$ to $\cos_p t$ and π_{pp} to π_p. When $p = q = 2$, the functions $\sin_{pq} t$, $\cos_{pq} t$ and π_{pq} are obviously reduced to the usual sin, cos and pi.

2 Properties of Generalized Trigonometric Functions

The functions $\sin_{pq} t, \cos_{pq} t$ satisfy the following proposition.

Proposition 2.1. For every $t \in [0, \pi_{pq}/2]$, the following equalities hold:

(i) $(\sin_{pq} t)' = \cos_{pq} t$, (ii) $(\cos_{pq} t)' = -\dfrac{q}{p}(\sin_{pq} t)^{q-1}(\cos_{pq} t)^{-p+2}$,

(iii) $(\cos_{pq} t)^p + (\sin_{pq} t)^q = 1$, (iv) $\cos_{pq} t = \left(\sin_{q^*p^*}\left(\dfrac{\pi_{q^*p^*}}{\pi_{pq}}\left(\dfrac{\pi_{pq}}{2} - t\right)\right)\right)^{p^*-1}$.

Proof. (i) By differentiating $y = \sin_{pq} t$ and by Eq.(1.3), we obtain that

$$\frac{dy}{dt} = \frac{1}{dt/dy} = \frac{1}{\left(1 - y^q\right)^{-1/p}} = \left(1 - (\sin_{pq} t)^q\right)^{1/p} = \cos_{pq} t.$$

(iii) By Eq.(1.3), the equality is obvious.
(ii) By differentiating both sides of (iii), the equality is proved.
(iv) By putting $x = \cos_{pq} t$ to (ii), we obtain that

$$\frac{dx}{dt} = -\frac{q}{p}\left(1 - x^p\right)^{1/q^*} x^{-p+2}.$$

By the method of separation of variables, we obtain that

$$\begin{aligned} \frac{\pi_{pq}}{2} - t =& \frac{p}{q}\int_0^x \left(1 - u^p\right)^{-1/q^*} u^{p-2} du \\ =& \frac{p^*}{q}\int_0^{x^{p-1}} \left(1 - v^{p^*}\right)^{-1/q^*} dv \quad \left(v = u^{p-1}\right) \\ =& \frac{p^*}{q}\sin^{-1}_{q^*p^*}\left(\left(\cos_{pq} t\right)^{p-1}\right). \end{aligned} \tag{2.1}$$

By Eq.(1.2), the equality is proved. □

3 Properties of Generalized Pi's

We know some equalities containing two generalized pi's as follows:

$$\text{(a)}\ \frac{\pi_{q^*,p^*}}{\pi_{p,q}} = \frac{q}{p^*}, \quad \text{(b)}\ \frac{\pi_{p^*,p}}{\pi_{2,p}} = 2^{-2/p+1}. \tag{3.1}$$

The first equality is already mentioned in Eq.(1.2), and the second can be proved directly from Legendre Duplication Formula. (It is already pointed out by Takeuchi [5].) In this paper, we give other relations containing two generalized pi's.

Theorem 3.1 ([2]). For every real number $p \in (1, \infty)$, the following equalities hold:

(i) $\dfrac{\pi_{2p^*,2p}}{\pi_{p^*,p}} = 2^{1/p-1}$, (ii) $\dfrac{\pi_{p^*,2p^*}}{\pi_{p,2p}} = (p-1)2^{2/p-1}$, (iii) $\dfrac{\pi_{2p^*,p^*}}{\pi_{2p,p}} = 2^{-2/p+1}$.

By putting values to the parameters, we can evaluate the equalities.

Example 3.2. (1) Putting $p = 3$ or $3/2$ gives

$$\frac{\pi_{3,6}}{\pi_{3/2,3}} = \frac{1}{2^{2/3}}, \ \frac{\pi_{6,3}}{\pi_{3,3/2}} = \frac{1}{2^{1/3}}. \tag{3.2}$$

(2) Putting $p = 4$ or $4/3$ gives

$$\frac{\pi_{8/3,8}}{\pi_{4/3,4}} = \frac{1}{2^{3/4}}, \ \frac{\pi_{8,8/3}}{\pi_{4,4/3}} = \frac{1}{2^{1/4}}, \ \frac{\pi_{4/3,8/3}}{\pi_{4,8}} = \frac{3}{2^{1/2}}, \ \frac{\pi_{8/3,4/3}}{\pi_{8,4}} = 2^{1/2}. \tag{3.3}$$

(3) Putting $p = 5$ or $5/4$ gives

$$\frac{\pi_{5/2,10}}{\pi_{5/4,5}} = \frac{1}{2^{4/5}}, \frac{\pi_{10,5/2}}{\pi_{5,5/4}} = \frac{1}{2^{1/5}}, \frac{\pi_{5/4,5/2}}{\pi_{5,10}} = \frac{4}{2^{3/5}}, \frac{\pi_{5/2,5/4}}{\pi_{10,5}} = 2^{3/5}. \tag{3.4}$$

The Beta function can be represented by three Gamma functions:

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{3.5}$$

For the Gamma function, there is a formula called Legendre Duplication Formula:

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma(z+1/2). \tag{3.6}$$

By applying Eq.(1.2) and (3.5) to (3.6), we can prove the second of (3.1) as follows:

$$\frac{\pi_{p^*,p}}{\pi_{2,p}} = \frac{\dfrac{2}{p}B\Big(\dfrac{1}{p}, \dfrac{1}{p}\Big)}{\dfrac{2}{p}B\Big(\dfrac{1}{2}, \dfrac{1}{p}\Big)} = \frac{\Gamma\Big(\dfrac{1}{p}\Big)\Gamma\Big(\dfrac{1}{p} + \dfrac{1}{2}\Big)}{\Gamma\Big(\dfrac{1}{2}\Big)\Gamma\Big(\dfrac{2}{p}\Big)} = 2^{-2/p+1}. \tag{3.7}$$

We prove Theorem 3.1 in a similar but somewhat complicated way. First, by combining two copies of (3.6), we deduce the following identity :

$$\frac{\Gamma(z)\Gamma(z+1/2)\Gamma(2w)}{\Gamma(w)\Gamma(w+1/2)\Gamma(2z)} = 2^{2(w-z)}. \tag{3.8}$$

By applying Eq.(3.5) to it, we can deduce many equalities containing two Beta functions such as the following lemma. However, all such equalities are equivalent to either (i) or (ii) of the Lemma 3.3. (The equality (iii) is a variant of (ii).)

Lemma 3.3. The following equalities hold:

(i) $\dfrac{B(1/2+x,x)}{B(2x,2x)} = 2^{2x}$, (ii) $\dfrac{B(2x,1/2-x)}{B(1-2x,x)} = 2^{4x-1}$,

(iii) $\dfrac{B(1/2+x,1-2x)}{B(1-x,2x)} = \dfrac{x}{1/2-x}2^{-4x+1}$.

Proof. (i) $z=x$, $w=2x$ into Eq.(3.8), we obtain that

$$\frac{B(1/2+x,x)}{B(2x,2x)} = \frac{\Gamma(x)\Gamma(x+1/2)\Gamma(4x)}{\Gamma(2x)\Gamma(2x+1/2)\Gamma(2x)} = 2^{2x}. \tag{3.9}$$

(ii) By putting $z=1/2-x$, $w=x$ into Eq.(3.8), we obtain that

$$\frac{B(2x,1/2-x)}{B(1-2x,x)} = \frac{\Gamma(1/2-x)\Gamma(1-x)\Gamma(2x)}{\Gamma(x)\Gamma(x+1/2)\Gamma(1-2x)} = 2^{4x-1}. \tag{3.10}$$

(iii) By using Eq.(3.10), we obtain that

$$\begin{aligned}\frac{B(1/2+x,1-2x)}{B(1-x,2x)} &= \frac{\Gamma(x+1)\Gamma(1-2x)\Gamma(x+1/2)}{\Gamma(3/2-x)\Gamma(1-x)\Gamma(2x)}\\ &= \frac{x}{1/2-x}.\frac{\Gamma(x)\Gamma(x+1/2)\Gamma(1-2x)}{\Gamma(1/2-x)\Gamma(1-x)\Gamma(2x)} = \frac{x}{1/2-x}2^{-4x+1}.\end{aligned} \tag{3.11}$$

□

Proof of Theorem 3.1. By putting $x=1/(2p)$ into the Lemma 3.3, we obtain that

(i) $$\frac{\pi_{2p^*,2p}}{\pi_{p^*,p}} = \frac{\dfrac{2}{2p}B\left(1-\dfrac{1}{2p^*},\dfrac{1}{2p}\right)}{\dfrac{2}{p}B\left(1-\dfrac{1}{p^*},\dfrac{1}{p}\right)} = \frac{B\left(\dfrac{1}{2}+\dfrac{1}{2p},\dfrac{1}{2p}\right)}{2B\left(\dfrac{1}{p},\dfrac{1}{p}\right)} = 2^{1/p-1}.$$

(ii) $$\frac{\pi_{p^*,2p^*}}{\pi_{p,2p}} = \frac{\dfrac{2}{2p^*}B\left(1-\dfrac{1}{p^*},\dfrac{1}{2p^*}\right)}{\dfrac{2}{2p}B\left(1-\dfrac{1}{p},\dfrac{1}{2p}\right)} = (p-1)\frac{B\left(\dfrac{1}{p},\dfrac{1}{2}-\dfrac{1}{2p}\right)}{B\left(1-\dfrac{1}{p},\dfrac{1}{2p}\right)} = (p-1)2^{2/p-1}.$$

(iii) $$\frac{\pi_{2p^*,p^*}}{\pi_{2p,p}} = \frac{\dfrac{2}{p^*}B\left(1-\dfrac{1}{2p^*},\dfrac{1}{p^*}\right)}{\dfrac{2}{p}B\left(1-\dfrac{1}{2p},\dfrac{1}{p}\right)} = (p-1)\frac{B\left(\dfrac{1}{2}+\dfrac{1}{2p},1-\dfrac{1}{p}\right)}{B\left(1-\dfrac{1}{2p},\dfrac{1}{p}\right)} = 2^{-2/p+1}.$$

□

4 Generalized Elliptic Integrals of Four Parameters.

4.1 Generalized Legendre Relation.

Takeuchi [6] defined generalized elliptic integrals of three parameters. In this paper, we define generalized elliptic integrals of four parameters as follows:

$$E_{pqrs}(k) = \int_0^1 \frac{(1-k^s t^q)^{1/r}}{(1-t^q)^{1/p}} dt, \ \ K_{pqrs}(k) = \int_0^1 \frac{(1-k^s t^q)^{1/r-1}}{(1-t^q)^{1/p}} dt.$$

Obviously, the following equation holds:

$$E_{pqrs}(0) = K_{pqrs}(0) = \int_0^1 \frac{1}{(1-t^q)^{1/p}} dt = \frac{\pi_{pq}}{2}. \tag{4.1}$$

The following formula is Generalized Legendre Relations of four parameters.

Theorem 4.1. Let $p \in (-\infty, 0) \cup (1, \infty)$, $\ q, r \in (1, \infty)$. For every $k \in [0, 1]$, we denote $k' = (1-k^s)^{1/s}$. Then the following equality holds:

$$E_{pqrs}(k)K_{prqs}(k') + K_{pqrs}(k)E_{prqs}(k') - K_{pqrs}(k)K_{prqs}(k') = \frac{\pi_{pq}\pi_{\sigma r}}{4}, \tag{4.2}$$

where $1/\sigma = 1/p - 1/q$.

Theorem 4.2. Under the same notations as in Theorem 4.1, The following differential equalities hold:

(i) $\displaystyle \frac{d}{dk} E_{pqrs}(k) = \frac{s/r}{k}\Big\{E_{pqrs}(k) - K_{pqrs}(k)\Big\},$

(ii) $\displaystyle \frac{d}{dk} K_{pqrs}(k) = \frac{s/\mu}{k(1-k^s)}\Big\{E_{pqrs}(k) - K_{pqrs}(k)\Big\} + \frac{(s/q)k^s}{k(1-k^s)} K_{pqrs}(k),$

(iii) $\displaystyle \frac{d}{dk} E_{pqrs}(k') = -\frac{(s/r)k^s}{k(1-k^s)}\Big\{E_{pqrs}(k') - K_{pqrs}(k')\Big\},$

(iv) $\displaystyle \frac{d}{dk} K_{pqrs}(k') = -\frac{s/\mu}{k(1-k^s)}\Big\{E_{pqrs}(k') - K_{pqrs}(k')\Big\} - \frac{s/q}{k} K_{pqrs}(k'),$

where $1/\mu = 1/q + 1/r - 1/p$.

Lemma 4.3. For every $k \in (0, 1)$, the following equality holds:

$$\int_0^1 \frac{(1-k^s t^q)^{1/r-2}}{(1-t^q)^{1/p}} dt = \frac{r^*/\mu}{1-k^s} E_{pqrs}(k) + \left\{\frac{r^*(k^s/q - 1/\mu)}{1-k^s} + 1\right\} K_{pqrs}(k). \tag{4.3}$$

where r^* denotes the Holder conjugate of r, that is, $1/r + 1/r^* = 1$.

Proof. We can calculate as follows:

$$
\begin{aligned}
&\frac{d}{dt}\big\{t(1-k^st^q)^{1/r-1}(1-t^q)^{1-1/p}\big\}\\
=&(1-t^q)^{-1/p}(1-k^st^q)^{1/r-2}\Big\{(1-k^st^q)(1-t^q)+\frac{q}{r^*}k^st^q(1-t^q)-\frac{q}{p^*}t^q(1-k^st^q)\Big\},\\
=&\frac{1}{k^s}(1-k^st^q)^{1/r-2}(1-t^q)^{-1/p}\Big\{\big(1-\frac{q}{r^*}+\frac{q}{p^*}\big)(1-k^st^q)^2+\big(k^s-\big(1-\frac{q}{r^*}+\frac{q}{p^*}\big)\\
&+\frac{q}{r^*}\big(1-k^s\big)\big)(1-k^st^q)-\frac{q}{r^*}\big(1-k^s\big)\Big\},\\
=&\frac{1}{k^s}\Big\{\frac{q}{\mu}(1-k^st^q)^{1/r}(1-t^q)^{-1/p}+\big(k^s-\frac{q}{\mu}+\frac{q}{r^*}(1-k^s)\big)(1-k^st^q)^{1/r-1}(1-t^q)^{-1/p}\\
&-\frac{q}{r^*}(1-k^s)(1-k^st^q)^{1/r-2}(1-t^q)^{-1/p}\Big\}.
\end{aligned}
\tag{4.4}
$$

By integrating both sides, we can prove the Lemma 4.3. □

Proof of Theorem 4.2. (i) By differentiating the definition of $E_{pqrs}(k)$, we obtain that

$$
\begin{aligned}
\frac{d}{dk}E_{pqrs}(k)&=\frac{s/r}{k}\int_0^1\frac{-k^st^q(1-k^st^q)^{1/r-1}}{(1-t^q)^{1/p}}dt\\
&=\frac{s/r}{k}\Big\{\int_0^1\frac{(1-k^st^q)^{1/r}}{(1-t^q)^{1/p}}dt-\int_0^1\frac{(1-k^st^q)^{1/r-1}}{(1-t^q)^{1/p}}dt\Big\}\\
&=\frac{s/r}{k}\{E_{pqrs}(k)-K_{pqrs}(k)\}.
\end{aligned}
$$

(ii) By differentiating the definition of $K_{pqrs}(k)$, we obtain that

$$
\begin{aligned}
\frac{d}{dk}K_{pqrs}(k)&=\frac{s/r^*}{k}\int_0^1\frac{k^st^q(1-k^st^q)^{1/r-2}}{(1-t^q)^{1/p}}dt\\
&=\frac{s/r^*}{k}\int_0^1\frac{(1-k^st^q)^{1/r-2}}{(1-t^q)^{1/p}}dt-\int_0^1\frac{(1-k^st^q)^{1/r-1}}{(1-t^q)^{1/p}}dt,
\end{aligned}
$$

by applying Eq.(4.3)

$$
=\frac{s/\mu}{k(1-k^s)}\{E_{pqrs}(k)-K_{pqrs}(k)\}+\frac{(s/q)k^s}{k(1-k^s)}K_{pqrs}(k). \tag{4.5}
$$

(iii) By using Eqs.(i), (ii), we can prove the (iii) as follows:

$$
\begin{aligned}
\frac{d}{dk}E'_{pqrs}(k)&=\frac{d}{dk'}E_{pqrs}(k')\cdot\frac{dk'}{dk}=\frac{s/r}{k'}\{E_{pqrs}(k')-K_{pqrs}(k')\}\Big(-\frac{k^{s-1}}{(k')^{s-1}}\Big),\\
&=-\frac{(s/r)k^s}{k(1-k^s)}\{E'_{pqrs}(k)-K'_{pqrs}(k)\}.
\end{aligned}
$$

$$
\begin{aligned}
\text{(iv)}\quad \frac{d}{dk}K_{pqrs}(k') &= \frac{d}{dk'}K_{pqrs}(k')\cdot\frac{dk'}{dk}\\
&= \Big\{\frac{s/\mu}{k'(1-(k')^s)}\{E_{pqrs}(k')-K_{pqrs}(k')\}+\frac{(s/q)(k')^r}{k'(1-(k')^s)}K_{pqrs}(k')\Big\}\Big(\frac{-k^{s-1}}{(k')^{s-1}}\Big)\\
&= -\frac{s/\mu}{k(1-k^s)}\{E_{pqrs}(k')-K_{pqrs}(k')\}-\frac{s/q}{k}K_{pqrs}(k').
\end{aligned}
$$

Hence, we have proved the Theorem 4.2. □

Proof Theorem 4.1. Denote $L(k)$ the left-hand side of (4.2). By using Eqs.(i), (ii), (iii) and (iv) of Theorem 4.2, we obtain that,

$$
\begin{aligned}
\frac{d}{dk}L(k) &= \frac{d}{dk}E_{pqrs}(k)\cdot K_{prqs}(k')+E_{pqrs}(k)\cdot\frac{d}{dk}K_{prqs}(k')+\frac{d}{dk}K_{pqrs}(k)\cdot E_{prqs}(k')\\
&\quad+K_{pqrs}(k)\cdot\frac{d}{dk}E_{prqs}(k')-\frac{d}{dk}K_{pqrs}(k)\cdot K_{prqs}(k')-K_{pqrs}(k)\cdot\frac{d}{dk}K_{prqs}(k')\\
&= \frac{s/r}{rk}\{E_{pqrs}(k)-K_{pqrs}(k)\}K_{prqs}(k')\\
&\quad-\frac{s/\mu}{k(1-k^s)}E_{pqrs}(k)\{E_{prqs}(k')-K_{prqs}(k')\}-\frac{(s/r)}{k}E_{pqrs}(k)K_{prqs}(k')\\
&\quad+\frac{s/\mu}{k(1-k^s)}\{E_{pqrs}(k)-K_{pqrs}(k)\}E_{prqs}(k')+\frac{(s/q)k^s}{k(1-k^s)}K_{pqrs}(k)E_{prqs}(k')\\
&\quad-\frac{(s/q)k^s}{k(1-k^s)}K_{pqrs}(k)\{E_{prqs}(k')-K_{prqs}(k')\}\\
&\quad-\frac{s/\mu}{k(1-k^s)}\{E_{pqrs}(k)-K_{pqrs}(k)\}K_{prqs}(k')-\frac{(s/q)k^s}{k(1-k^s)}K_{pqrs}(k)K_{prqs}(k')\\
&\quad+\frac{s/\mu}{k(1-k^s)}K_{pqrs}(k)\{E_{prqs}(k')-K_{prqs}(k')\}+\frac{(s/r)}{k}K_{pqrs}(k)K_{prqs}(k')=0. \qquad (4.6)
\end{aligned}
$$

So the function $L(k)$ is a constant.

We estimate the following two terms:

$$
\begin{aligned}
K_{pqrs}(k)-E_{pqrs}(k) &= \int_0^1\frac{k^st^q(1-k^st^q)^{1/r-1}}{(1-t^q)^{1/p}}dt\\
&\leqq \int_0^1\frac{k^s(1-k^s)^{1/r-1}}{(1-t^q)^{1/p}}dt=\frac{\pi_{pq}}{2}k^s(1-k^s)^{1/r-1},\\
K_{prqs}(k') &= \int_0^1\frac{\{1-(1-k^s)t^r\}^{1/q-1}}{(1-t^r)^{1/p}}dt\\
&\leqq \int_0^1\frac{k^{s(1/q-1)}}{(1-t^r)^{1/p}}dt=\frac{\pi_{pr}}{2}k^{s(1/q-1)}. \qquad (4.7)
\end{aligned}
$$

By above inequalities, we have that

$$0 \leqq \{K_{pqrs}(k) - E_{pqrs}(k)\}K_{prqs}(k') \leqq \frac{\pi_{pq}\pi_{pr}}{4}k^{s/q}(1-k^s)^{1/r-1}. \tag{4.8}$$

By the sandwich rule, the center term of Eq.(4.8) vanishes as $k \to 0$. Thus we obtain that

$$\begin{aligned}\lim_{k\to 0} L(k) &= \lim_{k\to 0}\{K_{pqrs}(k)E_{prqs}(k') - \{K_{pqrs}(k) - E_{pqrs}(k)\}K_{prqs}(k')\} \\ &= K_{pqrs}(0)E_{prqs}(1). \end{aligned} \tag{4.9}$$

Hence, we have proved the Theorem 4.1. □

4.2 Differential Equations

Theorem 4.4. The functions $y = K_{pqrs}(k)$ and $z = K_{pqrs}(k')$ respectively satisfy

$$k(1-k^s)\frac{d^2y}{dk^2} + \left\{\left(1+\frac{s}{q}-\frac{s}{p}\right) - \left(1+s+\frac{s}{q}-\frac{s}{r}\right)k^s\right\}\frac{dy}{dk} - \frac{s^2}{q}\left(1-\frac{1}{r}\right)k^{s-1}y = 0. \tag{4.10}$$

$$k(1-k^s)\frac{d^2z}{dk^2} + \left\{\left(1-\frac{s}{r}+\frac{s}{p}\right) - \left(1+s+\frac{s}{q}-\frac{s}{r}\right)k^s\right\}\frac{dz}{dk} - \frac{s^2}{q}\left(1-\frac{1}{r}\right)k^{s-1}z = 0. \tag{4.11}$$

Proof of Theorem 4.4. For Eq.(4.10), put $y = K_{pqrs}(k)$ and $x = E_{pqrs}(k)$ into (i) and (ii) of Theorem 4.2, we obtain that

$$\frac{dx}{dk} = \frac{s/r}{k}(x-y), \tag{4.12}$$

$$k(1-k^s)\frac{dy}{dk} = \frac{s}{\mu}(x-y) + \frac{s}{q}k^s y. \tag{4.13}$$

By differentiating both sides of Eq.(4.13), we obtain that

$$(1-(s+1)k^s)\frac{dy}{dk} + k(1-k^s)\frac{d^2y}{dk^2} = \frac{s}{\mu}\left(\frac{dx}{dk} - \frac{dy}{dk}\right) + \frac{s}{q}k^s\frac{dy}{dk} + \frac{s^2}{q}k^{s-1}y. \tag{4.14}$$

By putting Eq.(4.13) into (4.12), we have that

$$\frac{dx}{dk} = \frac{\mu}{r}(1-k^s)\frac{dy}{dk} - \frac{\mu s}{qr}k^{s-1}y. \tag{4.15}$$

By putting Eq.(4.15) into (4.14), we have proved Eq.(4.10).

Again for Eq.(4.11), put $z = K_{pqrs}(k')$ and $x = E_{pqrs}(k')$ into (iii) and (iv) of Theorem 4.2, we obtain that

$$\frac{dx}{dk} = -\frac{(s/r)k^s}{k(1-k^s)}(x-z), \tag{4.16}$$

$$k(1-k^s)\frac{dz}{dk} = -\frac{s}{\mu}(x-z) - \frac{s}{q}(1-k^s)z. \tag{4.17}$$

By differentiating both sides of Eq.(4.17) with respect to k, we obtain that

$$\big(1-(s+1)k^s\big)\frac{dz}{dk}+k(1-k^s)\frac{d^2z}{dk^2}=-\frac{s}{\mu}\Big(\frac{dx}{dk}-\frac{dz}{dk}\Big)+\frac{s^2}{q}k^{s-1}z-\frac{s}{q}(1-k^s)\frac{dz}{dk}. \tag{4.18}$$

By putting Eq.(4.17) into (4.16), we have that

$$\frac{dx}{dk}=\frac{\mu}{r}k^s\frac{dz}{dk}+\frac{\mu s}{qr}k^{s-1}z. \tag{4.19}$$

By putting Eq.(4.19) into (4.18), we have proved Eq.(4.11). □

Corollary 4.5. For every $p,s\in(1,\infty)$, the function $y=K_{ppps}(k)$ and $z=K_{ppps}(k')$ satisfy the following differential equation:

$$k(1-k^s)\frac{d^2y}{dk^2}+\big(1-(1+s)k^s\big)\frac{dy}{dk}-\frac{s^2(p-1)}{p^2}k^{s-1}y=0. \tag{4.20}$$

Remark. By putting $x=k^s$, we can translate (4.10) into the following equation:

$$x(1-x)\frac{d^2y}{dx^2}+\Big\{\Big(\frac{1}{p^*}+\frac{1}{q}\Big)-\Big(1+\frac{1}{q}+\frac{1}{r^*}\Big)x\Big\}\frac{dy}{dx}-\frac{1}{qr^*}y=0. \tag{4.21}$$

It is Gauss hyper-geometric equation. So we obtain that

$$K_{pqrs}(k)=\frac{\pi_{pq}}{2}F\Big(\frac{1}{q},\frac{1}{r^*},\frac{1}{p^*}+\frac{1}{q},k^s\Big), \tag{4.22}$$

where $F(a,b,c,x)$ is Gauss hyper-geometric function.

5 Similar Results to Salamin-Brent Formula

5.1 Similar Results to Gauss AGM Theorem

Gauss found an important formula concerning elliptic integrals and arithmetic-geometric mean.

Theorem 5.1 (Gauss). For $a_0 \geqq b_0 > 0$, we define two sequences $\{a_n\}$ and $\{b_n\}$ as follows:

$$a_{n+1}=\frac{a_n+b_n}{2},\ \ b_{n+1}=\sqrt{a_nb_n}. \tag{5.1}$$

Then two sequences $\{a_n\}$ and $\{b_n\}$ converge to the same limit. We denote it by $M_2(a_0,b_0)$ and call the arithmetic-geometric mean (AGM) of a_0 and b_0. Then the following formula holds:

$$\frac{1}{a_0}K_{2222}\Big(\frac{c_0}{a_0}\Big)=\frac{\pi/2}{M_2(a_0,b_0)}. \tag{5.2}$$

J. M. Borwein and P. B. Borwein [4] found two formulas which can give analogous results to Gauss's AGM Formula. Recently, Takeuchi [6], [7] made their theorems those of generalized pi's.

Theorem 5.2 (Borwein-Takeuchi). For $a_0 \geqq b_0 > 0$, we define three sequences $\{a_n\}, \{b_n\}$ and $\{c_n\}$ as follows:

$$a_{n+1} = \frac{a_n + 2b_n}{3},\ c_{n+1} = \frac{a_n - b_n}{3},\ b_n^3 = a_n^3 - c_n^3. \tag{5.3}$$

Then two sequences $\{a_n\}$ and $\{b_n\}$ converge to the same limit. We denote it by $M_3(a_0, b_0)$. Then the following formula holds:

$$\frac{1}{a_0} K_{3333}\Big(\frac{c_0}{a_0}\Big) = \frac{\pi_3/2}{M_3(a_0, b_0)}. \tag{5.4}$$

Theorem 5.3 (Borwein-Takeuchi). For $a_0 \geqq b_0 > 0$, we define three sequences $\{a_n\}, \{b_n\}$ and $\{c_n\}$ as follows:

$$a_{n+1} = \frac{a_n + 3b_n}{4},\ c_{n+1} = \frac{a_n - b_n}{4},\ b_n^2 = a_n^2 - c_n^2. \tag{5.5}$$

Then two sequences $\{a_n\}$ and $\{b_n\}$ converge to the same limit. We denote it by $A_4(a_0, b_0)$. Then the following formula holds:'

$$\frac{1}{\sqrt{a_0}} K_{4442}\Big(\frac{c_0}{a_0}\Big) = \frac{\pi_4/2}{\sqrt{A_4(a_0, b_0)}}. \tag{5.6}$$

Lemma 5.4. Two sequences $\{a_n\}$ and $\{b_n\}$ defined by Eq.(5.5) converge to the same limit.

Proof. From definition of $\{b_n\}$, we know that $b_n \leqq a_n$ for every n. From this inequality we have that

$$a_{n+1} - a_n = -\frac{3}{4}(a_n - b_n) \leqq 0. \tag{5.7}$$

and

$$b_{n+1}^2 - b_n^2 = a_{n+1}^2 - c_{n+1}^2 - b_n^2 = \frac{1}{2} b_n (a_n - b_n) \geqq 0. \tag{5.8}$$

So we obtain that

$$b_0 \leqq b_1 \leqq b_2 \leqq .... \leqq b_n \ \leqq \ a_n \leqq ... \leqq a_2 \leqq a_1 \leqq a_0 \tag{5.9}$$

Moreover, we can calculate as following :

$$0 \leqq a_{n+1} - b_{n+1} \leqq \frac{1}{4}(a_n - b_n). \tag{5.10}$$

By repeating Eq.(5.10), we obtain that

$$0 \leqq a_n - b_n \leqq \left(\frac{1}{4}\right)^n (a_0 - b_0). \tag{5.11}$$

By Sandwich rule, the center term of Eq.(5.11) converges to zero. By Nested Interval Theorem, two sequences $\{a_n\}$ and $\{b_n\}$ converge to the same limit. □

Lemma 5.5. For every integer k $(0 \leqq k \leqq n)$, we estimate that $c_n \leqq 8b_0\left(\dfrac{c_k}{8b_0}\right)^{2^{n-k}}$.

Proof. We can calculate as follows:

$$c_{n+1} = \frac{a_n - b_n}{4} = \frac{a_n^2 - b_n^2}{4(a_n + b_n)} \leqq \frac{c_n^2}{8b_0}. \tag{5.12}$$

By repeating Eq.(5.12), we can estimate c_n as follows:

$$c_n \leqq 8b_0\left(\frac{c_k}{8b_0}\right)^{2^{n-k}}.$$

Hence, we have proved Lemma 5.5. □

5.2 Salamin-Brent-Like Formula

In 1985-86, Salamin and Brent independently found a fast convergence formula for computing the value of π. The following is the Salamin-Brent Formula.

Theorem 5.6 (Salamin-Brent Formula). Let $a_0 = 1$ and $b_0 = 1/\sqrt{2}$, then

$$\pi = \frac{2\bigl(M_2\bigl(1, 1/\sqrt{2}\bigr)\bigr)^2}{\dfrac{1}{2} - \displaystyle\sum_{j=1}^{\infty} 2^j (a_j^2 - b_j^2)},$$

where $\{a_n\}$ and $\{b_n\}$ are the sequences defined by Eq.(5.1).

Recently, Takeuchi [5], [7] found two Salamin-Brent-like formulas for π_3 and π_4.

Theorem 5.7 (Takeuchi [5]). Let $a_0 = 1$ and $b_0 = 1/2^{1/3}$, then

$$\pi_3 = \frac{2\bigl(M_3\bigl(1, 1/2^{1/3}\bigr)\bigr)^2}{1 - 2\displaystyle\sum_{j=1}^{\infty} 3^j (a_j + c_j) c_j},$$

where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are the sequences defined by Eq.(5.3).

Theorem 5.8 (Takeuchi [7]). Let $a_0 = 1$ and $b_0 = 1/\sqrt{2}$, then

$$\pi_4 = \frac{2A_4\big(1, 1/\sqrt{2}\big)}{1 - \displaystyle\sum_{j=0}^{\infty} 2^j (a_j - b_j)},$$

where $\{a_n\}$ and $\{b_n\}$ are the sequences defined by Eq.(5.5).

The statement of the above result is different from that of Takeuchi [7]. However, it is the same. In the past few months, the author has tried to find Salamin-Brent-like formula for another π_{pq}. Unfortunately, I could not find it. However, I found a simpler proof of Theorem 5.8.

5.3 Proof of Theorem 5.8

To prove Theorem 5.8, we consider the case when $p = q = r = 4$, $s = 2$. That is,

$$\begin{aligned} K_{4442}(k) &= \int_0^1 (1 - k^2 t^4)^{-3/4} (1 - t^4)^{-1/4} dt, \\ E_{4442}(k) &= \int_0^1 (1 - k^2 t^4)^{1/4} (1 - t^4)^{-1/4} dt. \end{aligned} \tag{5.13}$$

From now, we abbreviate the suffices of K_{4442} and E_{4442}. By Theorem 4.1, we obtain the following corollary .

Corollary 5.9. For every $k \in (0, 1)$, we denote $k' = \sqrt{1 - k^2}$. Then the following equality holds:

$$E(k)K(k') + K(k)E(k') - K(k)K(k') = \frac{\pi_4}{2}.$$

Lemma 5.10. For every real number $k \in (0, 1)$, the following equalities hold:

(i) $K(k) = \dfrac{1}{\sqrt{1+3k}} K(m'),$ where $m = \dfrac{1-k}{1+3k}$.

(ii) $E(k) = \dfrac{\sqrt{1+3k}}{2} E(m') + \dfrac{1-k}{2\sqrt{1+3k}} K(m')$.

Proof. (i) Put $y = K(m')$. Then by Corollary 4.5, y satisfies the following equality:

$$m(1 - m^2)\frac{d^2y}{dm^2} + (1 - 3m^2)\frac{dy}{dm} - \frac{3}{4}my = 0. \tag{5.14}$$

Put $z = \dfrac{1}{(1+3k)^{1/2}} y$, that is $y = (1+3k)^{1/2} z$. Then we can calculate as follows:

$$\frac{dy}{dm} = \frac{d}{dk}\{(1+3k)^{1/2})z\} \cdot \frac{1}{dm/dk} = -\frac{1}{4}\Big\{(1+3k)^{5/2}\frac{dy}{dk} + \frac{3}{2}(1+3k)^{3/2} z\Big\}, \tag{5.15}$$

$$\frac{d^2y}{dm^2} = \frac{1}{16}(1+3k)^3\Big\{(1+3k)^{3/2}\frac{d^2z}{dk^2} + 9(1+3k)^{1/2}\frac{dz}{dk} + \frac{27}{4}(1+3k)^{-1/2} z\Big\}. \tag{5.16}$$

By substituting Eqs.(5.15), (5.16) into Eq.(5.14), we obtain that

$$\frac{1}{2}(1+3k)^{3/2}\Big\{k(1-k^2)\frac{d^2z}{dk^2} + (1-3k^2)\frac{dz}{dk} - \frac{3}{4}kz\Big\} = 0. \tag{5.17}$$

So $z = z(k)$ satisfies the above differential equation. By Corollary 4.5, there are two solution $K(k)$ and $K(k')$. Thus we can write down $z(k) = c_1 K(k) + c_2 K(k')$. Since we have that $z(0) = K(0) = \dfrac{\pi_4}{2}$ and $K(+0) = \infty$, we have that $c_1 = 1, c_2 = 0$. Hence we have proved (i).

(ii) By differentiating both sides of (i) with respect to k, we obtain that

$$\frac{d}{dk}K(k) = -\frac{3}{2}(1+3k)^{-3/2}K(m') + (1+3k)^{-1/2}\frac{d}{dk}K(m').k \tag{5.18}$$

By using Theorem 4.2 we obtain that

$$\begin{aligned}&\frac{1}{2k(1-k^2)}\{E(k) + (1-k^2)K(k)\}\\ &= -\frac{3}{2}(1+3k)^{-3/2}K(m') + \frac{1}{2k(1-k^2)}\Big\{\frac{(1+3k)^{1/2}}{2}E(m') - \frac{1-k^2}{2(1+3k)^{3/2}}K(m')\Big\}.\end{aligned}$$

By using (i), we have proved (ii). □

By putting $k = \dfrac{c_{n+1}}{a_{n+1}}$ into Lemma 5.10, we obtain lemma as following:

Lemma 5.11. If $\{a_n\}$ and $\{c_n\}$ are the sequences defined by Eq.(5.5), then the following equalities hold:

(i) $\dfrac{1}{\sqrt{a_{n+1}}}K\Big(\dfrac{c_{n+1}}{a_{n+1}}\Big) = \dfrac{1}{\sqrt{a_n}}K\Big(\dfrac{c_n}{a_n}\Big)$,

(ii) $2\sqrt{a_{n+1}}E\Big(\dfrac{c_{n+1}}{a_{n+1}}\Big) = \sqrt{a_n}E\Big(\dfrac{c_n}{a_n}\Big) + \dfrac{b_n}{\sqrt{a_n}}K\Big(\dfrac{c_n}{a_n}\Big)$.

Theorem 5.12. If $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are the sequences defined by Eq.(5.5) and $a_0 = a$, $b_0 = b$, $c_0 = c$. Then the following formula holds:

(i) $\dfrac{1}{\sqrt{a}}K\left(\dfrac{c}{a}\right) = \dfrac{\pi_4/2}{\sqrt{A_4(a,b)}}$,

(ii) $E\left(\dfrac{c}{a}\right) = \left\{1 - \dfrac{1}{a}\sum_{j=1}^{\infty} 2^j c_j\right\}K\left(\dfrac{c}{a}\right)$.

Proof. (i) By repeating Lemma (5.11-i), and taking limit $n \to \infty$, we obtain that

$$\frac{1}{\sqrt{a}}K\left(\frac{c}{a}\right) = \lim_{n\to\infty}\frac{1}{\sqrt{a_n}}K\left(\frac{c_n}{a_n}\right) = \frac{1}{\sqrt{A_4(a,b)}}K(0) = \frac{\pi_4/2}{\sqrt{A_4(a,b)}}. \tag{5.19}$$

(ii) By using Lemma (5.11-ii), we obtain that

$$\begin{aligned}
&\sqrt{a_0}\left\{E\left(\frac{c_0}{a_0}\right) - K\left(\frac{c_0}{a_0}\right)\right\}\\
&\quad= 2\sqrt{a_1}\left\{E\left(\frac{c_1}{a_1}\right) - K\left(\frac{c_1}{a_1}\right)\right\} + \left\{2\frac{a_1}{\sqrt{a_0}} - \sqrt{a_0} - \frac{b_0}{\sqrt{a_0}}\right\}K\left(\frac{c_0}{a_0}\right)\\
&\quad= 2\sqrt{a_1}\left\{E\left(\frac{c_1}{a_1}\right) - K\left(\frac{c_1}{a_1}\right)\right\} - 2\frac{c_1}{\sqrt{a_0}}K\left(\frac{c_0}{a_0}\right).
\end{aligned}$$

By repeating above equation, we obtain that

$$\begin{aligned}
&\sqrt{a}\left\{E\left(\frac{c}{a}\right) - K\left(\frac{c}{a}\right)\right\}\\
&\quad= 2^n\sqrt{a_n}\left\{E\left(\frac{c_n}{a_n}\right) - K\left(\frac{c_n}{a_n}\right)\right\} - \frac{1}{\sqrt{a}}\sum_{j=1}^{n} 2^j c_j K\left(\frac{c}{a}\right).
\end{aligned}$$

On the other hand, we have that

$$\begin{aligned}
&2^n\sqrt{a_n}\left\{K\left(\frac{c_n}{a_n}\right) - E\left(\frac{c_n}{a_n}\right)\right\}\\
&\qquad= 2^n\sqrt{a_n}\int_0^1 \frac{(c_n/a_n)^2 t^4}{(1-t^4)^{1/4}(1-(c_n/a_n)^2t^4)^{3/4}}dt\\
&\qquad\leqq \frac{2^n c_n^2}{b_n^{3/2}}\int_0^1 (1-t^4)^{-1/4}dt = \frac{\pi_4 2^n c_n^2}{2b_n^{3/2}}.
\end{aligned} \tag{5.20}$$

By the above equality, we have that

$$0 \leqq 2^n\sqrt{a_n}\left\{K\left(\frac{c_n}{a_n}\right) - E\left(\frac{c_n}{a_n}\right)\right\} \leqq \frac{\pi_4 2^n c_n^2}{2b_n^{3/2}}. \tag{5.21}$$

By Lemma 5.5, we obtain that

$$\frac{\pi_4 2^n c_n^2}{2b_n^{3/2}} \leqq \pi_4 2^5 b_0^{1/2} 2^n \Big(\frac{c_k}{8b_0}\Big)^{2^{n-k+1}}. \tag{5.22}$$

If we take k sufficiently large, we have that $\dfrac{c_k}{8b_0} \leqq \dfrac{1}{8}$. If we take n sufficiently large, we have that $2^{n-k+1} \geqq n$. So the right-hand side of Eq.(5.22) can be estimated from above by $\pi_4 b_0^{1/2} 2^{-2n+5}$. By Sandwich Rule, $2^n\sqrt{a_n}\Big\{K\Big(\dfrac{c_n}{a_n}\Big) - E\Big(\dfrac{c_n}{a_n}\Big)\Big\}$ vanishes when $n \to \infty$. Hence the formula (ii) is proved. □

Proof of Theorem 5.8. By putting $k = c/a = 1/\sqrt{2}$, we have that $k' = 1/\sqrt{2}$. By putting them into Corollary 5.9 we obtain that

$$2K\Big(\frac{1}{\sqrt{2}}\Big)E\Big(\frac{1}{\sqrt{2}}\Big) - \Big(K\Big(\frac{1}{\sqrt{2}}\Big)\Big)^2 = \frac{\pi_4}{2}. \tag{5.23}$$

By Theorem 5.12, we obtain that

$$K\Big(\frac{1}{\sqrt{2}}\Big) = \frac{\pi_4/2}{\sqrt{A_4\Big(1, \dfrac{1}{\sqrt{2}}\Big)}},\ E\Big(\frac{1}{\sqrt{2}}\Big) = \Big\{1 - \sum_{j=1}^{\infty} 2^j c_j\Big\}K\Big(\frac{1}{\sqrt{2}}\Big). \tag{5.24}$$

By substituting Eq.(5.24) into (5.23), we obtain that

$$\Big\{1 - 2\sum_{j=1}^{\infty} 2^j c_j\Big\}K\Big(\frac{1}{\sqrt{2}}\Big)^2 = \frac{\pi_4}{2}.$$

So Theorem 5.8 is proved. □

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