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Relations between Two Generalized Pi's

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1 Introduction

For real numbers $p, q \in (1, \infty)$, we introduce generalized pi's as follows:

$$\pi_{p,q} = 2\int_0^1 (1-t^q)^{-1/p}\,dt = \frac{2}{q}B\Big(\frac{1}{p^*}, \frac{1}{q}\Big), \quad \text{where } \frac{1}{p^*} = 1 - \frac{1}{p}. \tag{1.1}$$

Here, $B(\cdot,\cdot)$ denotes the Beta function. Clearly, π_{22} agrees with the usual π. The reader can study the generalized pi's by a book of Lang and Edmunds [1]. There are some equalities containing two generalized pi's, for example,

$$\text{(a)}\ \frac{\pi_{q^*,\,p^*}}{\pi_{p,q}} = \frac{q}{p^*}, \qquad \text{(b)}\ \frac{\pi_{p^*,\,p}}{\pi_{2,p}} = 2^{-2/p}. \tag{1.2}$$

Eq. (1.2-a) can be derived directly from Eq. (1.1). Eq. (1.2-b) is given in [3], and it can be proved directly from Legendre Duplication Formula. See Section 2.

In the paper, we give other relations containing two generalized pi's.

Theorem. For every real number $p \in (1, \infty)$, the following equalities hold:

(i) $\dfrac{\pi_{2p^*,\,2p}}{\pi_{p^*,\,p}} = 2^{1/p-1}$, (ii) $\dfrac{\pi_{p^*,\,2p^*}}{\pi_{p,\,2p}} = (p-1)2^{2/p-1}$, (iii) $\dfrac{\pi_{2p^*,\,p^*}}{\pi_{2p,\,p}} = 2^{-2/p+1}$.

By putting values to the parameter, we can evaluate the equalities.

Example. (1) Putting $p = 3$ or $3/2$ gives $\dfrac{\pi_{3,6}}{\pi_{3/2,3}} = \dfrac{1}{2^{2/3}}$, $\dfrac{\pi_{6,3}}{\pi_{3,3/2}} = \dfrac{1}{2^{1/3}}$.

(2) Putting $p = 4$ or $4/3$ gives $\dfrac{\pi_{8/3,8}}{\pi_{4/3,4}} = \dfrac{1}{2^{3/4}}$, $\dfrac{\pi_{8,8/3}}{\pi_{4,4/3}} = \dfrac{1}{2^{1/4}}$, $\dfrac{\pi_{4/3,8/3}}{\pi_{4,8}} = \dfrac{3}{2^{1/2}}$, $\dfrac{\pi_{8/3,4/3}}{\pi_{8,4}} = 2^{1/2}$.

(3) Putting $p = 5$ or $5/4$ gives $\dfrac{\pi_{5/2,10}}{\pi_{5/4,5}} = \dfrac{1}{2^{4/5}}$, $\dfrac{\pi_{10,5/2}}{\pi_{5,5/4}} = \dfrac{1}{2^{1/5}}$, $\dfrac{\pi_{5/4,5/2}}{\pi_{5,10}} = \dfrac{4}{2^{3/5}}$, $\dfrac{\pi_{5/2,5/4}}{\pi_{10,5}} = 2^{3/5}$.

2 Proof of the Theorem

The Beta function can be represented by three Gamma functions:

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{2.1}$$

For the Gamma function, there is a formula called Legendre Duplication Formula:

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma(z+1/2). \tag{2.2}$$

For the proof, see [2], for example. By applying Eq. (2.1) to it, we can prove Eq. (1.2-b) as follows:

$$\frac{\pi_{p^*,p}}{\pi_{2,p}} = \frac{\frac{2}{p}B\Big(\frac{1}{p},\frac{1}{p}\Big)}{\frac{2}{p}B\Big(\frac{1}{2},\frac{1}{p}\Big)} = \frac{\Gamma\Big(\frac{1}{p}\Big)\Gamma\Big(\frac{1}{p}+\frac{1}{2}\Big)}{\Gamma\Big(\frac{1}{2}\Big)\Gamma\Big(\frac{2}{p}\Big)} = 2^{-2/p+1}. \tag{2.3}$$

We prove our theorem in a similar but somewhat complicated way. First, by combining two copies of Eq. (2.2), we deduce the following identity:

$$\frac{\Gamma(z)\Gamma(z+1/2)\Gamma(2w)}{\Gamma(w)\Gamma(w+1/2)\Gamma(2z)} = 2^{2(w-z)}. \tag{2.4}$$

By applying Eq. (2.1) to it, we can deduce many identities containing two Beta functions. However, all such identities are equivalent to either (i) or (ii) of the following lemma. (The identity (iii) is a variant of (ii).)

Lemma. The following qualities hold:

(i) $\dfrac{B(1/2+x,x)}{B(2x,2x)} = 2^{2x}$, (ii) $\dfrac{B(2x,1/2-x)}{B(1-2x,x)} = 2^{4x-1}$, (iii) $\dfrac{B(1/2+x,1-2x)}{B(1-x,2x)} = \dfrac{x}{1/2-x}2^{-4x+1}$.

Proof. (i) By putting $z = x$, $w = 2x$ into Eq. (2.4), we obtain that

$$\frac{B(1/2+x,x)}{B(2x,2x)} = \frac{\Gamma(x)\Gamma(x+1/2)\Gamma(4x)}{\Gamma(2x)\Gamma(2x+1/2)\Gamma(2x)} = 2^{2x}. \tag{2.5}$$

(ii) By putting $z = 1/2-x$, $w = x$ into Eq. (2.4), we obtain that

$$\frac{B(2x,1/2-x)}{B(1-2x,x)} = \frac{\Gamma(1/2-x)\Gamma(1-x)\Gamma(2x)}{\Gamma(x)\Gamma(x+1/2)\Gamma(1-2x)} = 2^{4x-1}. \tag{2.6}$$

(iii) By using Eq. (2.6), we obtain that

$$\begin{aligned}\frac{B(1/2+x,1-2x)}{B(1-x,2x)} &= \frac{\Gamma(x+1)\Gamma(1-2x)\Gamma(x+1/2)}{\Gamma(3/2-x)\Gamma(1-x)\Gamma(2x)} \\ &= \frac{x}{1/2-x}\cdot\frac{\Gamma(x)\Gamma(x+1/2)\Gamma(1-2x)}{\Gamma(1/2-x)\Gamma(1-x)\Gamma(2x)} = \frac{x}{1/2-x}2^{-4x+1}.\end{aligned} \tag{2.7}$$

Proof of the Theorem. By putting $x = 1/(2p)$ into the lemma, we obtain that

$$\frac{\pi_{2p^*,2p}}{\pi_{p^*,p}} = \frac{\frac{2}{2p}B\Big(1-\frac{1}{2p^*},\frac{1}{2p}\Big)}{\frac{2}{p}B\Big(1-\frac{1}{p^*},\frac{1}{p}\Big)} = \frac{B\Big(\frac{1}{2}+\frac{1}{2p},\frac{1}{2p}\Big)}{2B\Big(\frac{1}{p},\frac{1}{p}\Big)} = 2^{1/p-1}. \tag{2.8}$$

$$\frac{\pi_{p^*,2p^*}}{\pi_{p,2p}} = \frac{\frac{2}{2p^*}B\Big(1-\frac{1}{p^*},\frac{1}{2p^*}\Big)}{\frac{2}{2p}B\Big(1-\frac{1}{p},\frac{1}{2p}\Big)} = (p-1)\frac{B\Big(\frac{1}{p},\frac{1}{2}-\frac{1}{2p}\Big)}{B\Big(1-\frac{1}{p},\frac{1}{2p}\Big)} = (p-1)2^{2/p-1}. \tag{2.9}$$

$$\frac{\pi_{2p^*,p^*}}{\pi_{2p,p}} = \frac{\frac{2}{p^*}B\Big(1-\frac{1}{2p^*},\frac{1}{p^*}\Big)}{\frac{2}{p}B\Big(1-\frac{1}{2p},\frac{1}{p}\Big)} = (p-1)\frac{B\Big(\frac{1}{2}+\frac{1}{2p},1-\frac{1}{p}\Big)}{B\Big(1-\frac{1}{2p},\frac{1}{p}\Big)} = 2^{-2/p+1}. \tag{2.10}$$

3 Comments on Generalized Trigonometric Functions

For real numbers $p, q \in (1, \infty)$, we define generalized sine $y = \sin_{p,q} x$ as the inverse of the following function:

$$\sin_{p,q}^{-1} y = \int_0^y (1 - t^q)^{-1/p} \, dt, \quad \text{where } y \in [-1, 1]. \tag{3.1}$$

The domain of variable is extended to whole the real numbers by $\sin_{p,q} \big(x + \pi_{p,q}\big) = -\sin_{p,q} x$. We define generalized cosine by $\cos_{p,q} x = \big(\sin_{p,q} x\big)'$, the derivative. Recently, Takeuchi [3] gave the following formulas:

$$\sin_{2,p} \big(2^{2/p} x\big) = 2^{2/p} \big(\sin_{p^*,p} x\big) \big(\cos_{p^*,p} x\big)^{p^*-1}, \tag{3.2}$$

$$\cos_{2,p} \big(2^{2/p} x\big) = \big(\cos_{p^*,p} x\big)^{p^*} - \big(\sin_{p^*,p} x\big)^{p}. \tag{3.3}$$

He also gave Eq. (1.2-b) as a direct corollary of the above formulas. The authors consider that there are formulas of generalized trigonometric functions which correspond to our theorem.

References

[1] J. Lang and D. E. Edmunds, Eigenvalues, Embeddings and Generalized Trigonometric Functions, Lecture Notes in Math., Vol. 2016, Springer, 2011.

[2] A. F. Nikiforov and V. B. Uvarov, Special Functions of Mathematical Physics, Birkhäuser, 1988.

[3] S. Takeuchi, Multiple-angle formulas of generalized trigonometric functions with two parameters, J. Math. Anal. Appl. **444** (2016), 1000–1014.

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