Application of generalised Riccati equations to analysis of asymptotic forms of solutions of perturbed half-linear ordinary differential equations

Sokea Luey*

Graduate School of Engineering,
Gifu University,
Gifu, 501-1193, Japan
Email: lueysokea2013.sl@gmail.com
*Corresponding author

Hiroyuki Usami

Faculty of Engineering,
Gifu University,
Gifu, 501-1193, Japan
Email: husami@gifu-u.ac.jp

Abstract: Asymptotic forms of solutions of half-linear ordinary differential equation $\left(|u'|^{\alpha-1}u'\right)' = \alpha\left(1 + b(t)\right)|u|^{\alpha-1}u$ are investigated under smallness conditions on $b(t)$. The proof of the main result is based on analysis of solutions of generalised Riccati equations related to this half-linear equation.

Keywords: half-linear ordinary differential equation; asymptotic form; Riccati equation; positive solution; perturbation theory; asymptotic behaviour.

Reference to this paper should be made as follows: Luey, S. and Usami, H. (2021) 'Application of generalised Riccati equations to analysis of asymptotic forms of solutions of perturbed half-linear ordinary differential equations', *Int. J. Dynamical Systems and Differential Equations*, Vol. 11, Nos. 3/4, pp.378–390.

Biographical notes: Sokea Luey has got a Master's degree in 2019 at Aichi University of Education, Japan. She had studied generalised trigonometric functions. She now studies asymptotic theory of half-linear ordinary differential equations in order to get a doctor's degree.

Hiroyuki Usami is a Mathematics Professor of Gifu University who is the supervisor of Sokea Luey. His interest is mainly asymptotic theory of various classes of differential equations.

1 Introduction

Let us consider the quasilinear ordinary differential equations of the following type near $+\infty$:

$$\big(|u'|^{\alpha-1}u'\big)' = \alpha\big(1+b(t)\big)|u|^{\alpha-1}u. \tag{1.1}$$

Here we always assume that $\alpha > 0$ is a constant and $b \in C[0,\infty)$. A C^1-function u defined near $+\infty$ is called a solution of equation (1.1) if $|u'|^{\alpha-1}u'$ is of class C^1, and (1.1) is satisfied for all sufficiently large t.

When $\alpha = 1$, equation (1.1) reduces to the linear equation

$$u'' = \big(1+b(t)\big)u. \tag{1.2}$$

Therefore, equations of the type (1.1) can be regarded as generalisations of second order linear equations. In fact, for a solution u of equation (1.1) and a constant $C,\ Cu$ is also a solution of equation (1.1); however, the sum of two solutions of equation (1.1) is not always a solution of equation (1.1). Thus equations of such types are often called half-linear equations.

Our aim of the paper is to study the following Problem:

Problem. When $b(t)$ is sufficiently small near $+\infty$, in some sense, what are the asymptotic forms of solutions of equation (1.1)?

For the case where $\alpha = 1$, that is, for equation (1.2) such a problem has been extensively investigated, see Bodine and Lutz (2015), Copple (1965). For example, it has been well known in Copple (1965) and Hartman (1982) that every nontrivial solution u of equation (1.2) satisfies

$$u(t) \sim ce^t \text{ or } \quad u(t) \sim ce^{-t} \text{ as } t \to \infty$$

for some constant $c \neq 0$ if $\int^\infty |b(t)|\mathrm{d}t < \infty$. Here, for functions $f(t)$ and $g(t)$ defined near ∞ we write "$f(t) \sim g(t)$ as $t \to \infty$" if $\lim_{t\to\infty} f(t)/g(t) = 1$.

We make an attempt to extend this simple result to half-linear equation (1.1).

In the authors' previous paper Luey and Usami (2021), we have established the following result as an answer to the Problem:

Theorem 1.0: *Suppose that one of the next conditions holds:*

- $b(t) \geq 0$ *near* $+\infty$ *and* $\int^\infty b(t)\mathrm{d}t < \infty$;
- $-1 < b(t) \leq 0$ *near* $+\infty$ *and* $\int^\infty \big[-b(t)\big]\mathrm{d}t < \infty$.

Then, every nontrivial solution u *of equation* (1.1) *has the asymptotic form*

$$u(t) \sim ce^t \text{ or } \ u(t) \sim ce^{-t} \text{ as } t \to \infty$$

for some constant $c \neq 0$.

In the authors' previous paper the signum conditions on $b(t)$, as stated above, are essentially used to prove Theorem 1.0. However, we conjecture that the signum conditions on $b(t)$ may be superfluous. In the present paper, we show that the signum conditions on $b(t)$ can be removed from the assumption of Theorem 1.0 if another smallness condition is imposed on $b(t)$. The following, which gives a partial improvement of Theorem 1.0, is the main result of the paper:

Theorem 1.1: *Suppose that*

$$\lim_{t\to\infty} b(t) = 0 \tag{1.3}$$

and

$$\int^{\infty} |b(t)| \mathrm{d}t < \infty. \tag{1.4}$$

Then every nontrivial solution u *of equation* (1.1) *has the asymptotic form*

$$u(t) \sim ce^t \text{ or } \ u(t) \sim ce^{-t} \ as \ t \to \infty$$

for some constant $c \neq 0$.

This paper is organised as follows. In Section 2 we state several preparatory results which are employed in proving Theorem 1.1. In Section 3 the proof of Theorem 1.1 will be given. The proof is based on asymptotic analysis of generalised Riccati equations associated with equation (1.1). Related results concerning generalised Riccati equations are found in Došlý and Řehák (2005) and Pátíková (2008).

Remark 1.2. (i) When $\alpha = 1$, Theorem 1.1 was introduced, for example, in Bodine and Lutz (2015), Copple (1965), and Hartman (1982). In this case condition (1.3) need not be assumed. So we also conjecture that Theorem 1.1 is still valid even if (1.3) is dropped from the assumption. To prove this fact is the theme of our future study.
(ii) When $b(t) \equiv 0$, Theorem 1.1 was introduced in Došlý and Řehák (2005).

2 Preparatory results

In this section we give several lemmas employed later.

Lemma 2.1: *Let* (1.3) *and* (1.4) *hold. Then every nontrivial solution of equation* (1.1) *is of constant sign near* $+\infty$.

Proof: Let $T > 0$ be a sufficiently large number such that $1 + b(t) > 0$ for $t \geq T$, and $u(t)$ be a nontrivial solution of equation (1.1) on $[T, \infty)$. If u is not of constant sign near $+\infty$, then there are two points $T_1, T_2 \geq T$ satisfying $T_1 < T_2$ and

$$u(T_1) = 0, u'(T_1) \geq 0, u'(T_2) = 0, \text{ and } u(t) > 0 \text{ for } t \in (T_1, T_2).$$

An integration of equation (1.1) on $[T_1, T_2]$ gives

$$-\big[u'(T_1)\big]^{\alpha} = \alpha \int_{T_1}^{T_2} \big(1 + b(s)\big) u(s)^{\alpha} \mathrm{d}s,$$

which is an obvious contradiction. So u is of constant sign near $+\infty$. This completes the proof. □

Lemma 2.2: *Let* (1.3) *and* (1.4) *hold. Then every nontrivial solution* u *of equation* (1.1) *satisfies one of the following two properties near* $+\infty$:

(i) $|u'(t)| \uparrow \infty$ *(and therefore,* $|u(t)| \uparrow \infty$*)* *as* $t \to \infty$;

(ii) $|u'(t)|, \ |u(t)| \downarrow 0$ *as* $t \to \infty$.

Since $u(t)$ is a solution of equation (1.1) if and only if so is $-u(t)$, in this paper we will consider mainly eventually positive solutions.

Proof of Lemma 2.2: Let $T > 0$ be a sufficiently large number such that $1 + b(t) > 0, \ t \geq T$. We may suppose that $u(t) > 0, \ t \geq T$. Then, by equation (1.1) we see that $|u'(t)|^{\alpha-1} u'(t)$ is increasing on $[T, \infty)$; that is, $u'(t)$ is increasing on $[T, \infty)$. We divide the argument into several cases by the limit of $u'(t)$ as $t \to \infty$.

Let $u'(t) \uparrow \infty$ as $t \to \infty$. Then property (i) of the statement holds.

Next, let $u'(t) \uparrow c$ as $t \to \infty$ for some constant $c > 0$. Then $u(t) \sim ct$ as $t \to \infty$, and an integration of equation (1.1) gives

$$\begin{aligned} |u'(t)|^{\alpha-1} u'(t) - |u'(T)|^{\alpha-1} u'(T) &= \alpha \int_T^t \big(1 + b(s)\big) u(s)^{\alpha} \mathrm{d}s \\ &\geq c_1 \int_T^t \big(1 + b(s)\big) s^{\alpha} \mathrm{d}s \end{aligned} \tag{2.1}$$

for some constant $c_1 > 0$. Since

$$\begin{aligned} \int_T^t \big(1 + b(s)\big) s^{\alpha} \mathrm{d}s \geq \frac{1}{\alpha+1} \big(t^{\alpha+1} - T^{\alpha+1}\big) - t^{\alpha} \int_T^{\infty} |b(s)| \mathrm{d}s \\ \longrightarrow \infty \text{ as } t \to \infty, \end{aligned}$$

(2.1) is a contradiction to the fact $\lim_{t\to\infty} u'(t) = c$.

Let $u'(t) \uparrow 0$ as $t \to \infty$. Then $u'(t) < 0$ near $+\infty$; and therefore $u(t)$ decreases near $+\infty$. Since $u(t) > 0$, we have $u(t) \downarrow l$ for some constant $l \geq 0$. If $l > 0$, then $u(t) \geq l$ near $+\infty$. We get from (2.1)

$$\begin{aligned} |u'(t)|^{\alpha-1} u'(t) - |u'(T)|^{\alpha-1} u'(T) &\geq l^{\alpha} \int_T^t \big(1 + b(s)\big) \mathrm{d}s \\ &\geq l^{\alpha} \Big\{ (t - T) - \int_T^{\infty} \big|b(s)\big| \mathrm{d}s \Big\} \\ &\longrightarrow \infty \text{ as } t \to \infty. \end{aligned}$$

This is a contradiction to the fact $\lim_{t\to\infty} u'(t) = 0$. Therefore $u(t) \downarrow 0$ (and $u'(t) \uparrow 0$), and property (ii) holds.

Finally let $u'(t) \uparrow c$ as $t \to \infty$ for some constant $c < 0$. However, this implies that $u(t) \sim ct$ as $t \to \infty$. Since $u(t) > 0$, this is an obvious contradiction.

This completed the proof. □

3 Proof of the main result

In this section we give the proof of our main result Theorem 1.1. Recall that every nontrivial solution u of equation (1.1) satisfies either (i) or (ii) of Lemma 2.2. We consider asymptotic forms of solutions of these two types separately. More precisely, in Section 3.1 we determine the asymptotic form of nontrivial solutions of equation (1.1) satisfying (i) of Lemma 2.2, while in Section 3.2 we determine that of nontrivial solutions of equation (1.1) satisfying (ii) of Lemma 2.2. The proof of Theorem 1.1 will be completed immediately by unifying these results. Note that we always assume (1.3) and (1.4): $\lim_{t\to\infty} b(t) = 0$ and $\int^{\infty} |b(t)|\mathrm{d}t < \infty$.

3.1 Asymptotic form of nontrivial increasing solutions

We consider asymptotic forms of nontrivial solutions $u(t)$ satisfying property (i) of Lemma 2.2: $|u'(t)|,\ |u(t)| \uparrow \infty$ as $t \to \infty$.

Proposition 3.1: *Every nontrivial solution u of* (1.1) *satisfying property* (i) *of Lemma 2.2 has the asymptotic form*

$$u(t) \sim ce^t \text{ as } t \to \infty,$$

for some constant $c \neq 0$.

The proof of Proposition 3.1 needs several lemmas.

Lemma 3.2: *Let u be a positive solution of* (1.1) *satisfying property* (i) *of Lemma 2.2, and put $w = \big(u'/u\big)^{\alpha}$ for sufficiently large t. Then w satisfies the generalised Riccati equation*

$$w' = \alpha\big(1 + b(t)\big) - \alpha w^{\frac{\alpha+1}{\alpha}}. \tag{3.1}$$

This lemma can be proved by a direct computation.

Lemma 3.3: *Let u be a positive solution of* (1.1) *satisfying property* (i) *of Lemma 2.2. Then $\lim_{t\to\infty} u'(t)/u(t) = 1$.*

Proof: Put $p(t) = \big(1 + b(t)\big)^{\alpha/(\alpha+1)}$. Then $\lim_{t\to\infty} p(t) = 1$, and the function $w = \big(u'/u\big)^{\alpha}$ satisfies

$$w' = \alpha\Big(p(t)^{\frac{\alpha+1}{\alpha}} - w^{\frac{\alpha+1}{\alpha}}\Big) \tag{3.2}$$

by Lemma 3.2. It is sufficient to show that $\lim_{t\to\infty} w(t) = 1$. We consider the following three exclusive cases separately:

Case (a): $w(t) \geq p(t)$ near $+\infty$;

Case (b): $w(t) \leq p(t)$ near $+\infty$;

Case (c): $w(t) - p(t)$ changes the sign in any neighbourhood of $+\infty$.

Let *Case* (a) occur. By (3.2) we have $w'(t) \leq 0$; so $w(t)$ decreases near $+\infty$. Since $w(t) \geq p(t)$ and $\lim_{t\to\infty} p(t) = 1$, there is a limit $\lim_{t\to\infty} w(t) = L \in [1, \infty)$. Let $t \to \infty$ in (3.2). Then we have $\lim_{t\to\infty} w'(t) = \alpha\big(1 - L^{(\alpha+1)/\alpha}\big)$. Since $w(t)$ is bounded, $\lim_{t\to\infty} w'(t)$ must be 0; which means that $L = 1$. So $\lim_{t\to\infty} w(t) = 1$.

Let *Case* (b) occur. We can show that $\lim_{t\to\infty} w(t) = 1$ similarly.

Finally let *Case* (c) occur. Put $\underline{L} = \liminf_{t\to\infty} w(t)$ and $\overline{L} = \limsup_{t\to\infty} w(t)$. Note that $w'(t) > 0$ [resp. $w'(t) < 0$] if and only if $w(t) < p(t)$ [resp. $w(t) > p(t)$]. Therefore $0 < \underline{L} \leq \overline{L} < \infty$.

To prove $\lim_{t\to\infty} w(t) = 1$, that is $\underline{L} = \overline{L} = 1$, we suppose the contrary that this is not the case.

If $\underline{L} = \overline{L}$, then we can show $\underline{L} = \overline{L} = 1$ as before. So we may assume $\underline{L} < \overline{L}$. From (3.2) and the fact that $0 < \underline{L} < \overline{L} < \infty$ we have $\underline{L} \leq 1 \leq \overline{L}$ (and $\underline{L} < \overline{L}$). Consequently there are three possibilities:

Case (c)-(i): $\underline{L} < 1 < \overline{L}$;

Case (c)-(ii): $\underline{L} < 1 = \overline{L}$;

Case (c)-(iii): $\underline{L} = 1 < \overline{L}$.

Let *Case* (c)-(i) hold. Put $\underline{L} = 1 - \delta\ (0 < \delta < 1)$. Then there is a sequence $\{t_n\}$ satisfying

$$t_1 < t_2 < \cdots < t_n < t_{n+1} < \cdots;\ \lim_{n\to\infty} t_n = \infty;$$
$$w'(t_n) = 0, \quad \text{and} \quad w(t_n) < 1 - (\delta/2), \quad n \in \mathbb{N}.$$

By putting $t = t_n$ in (3.2), we get,

$$\begin{aligned} 0 = w'(t_n) &= \alpha\Big[p(t_n)^{\frac{\alpha+1}{\alpha}} - w(t_n)^{\frac{\alpha+1}{\alpha}}\Big] \\ &> \alpha\Big[p(t_n)^{\frac{\alpha+1}{\alpha}} - \Big(1 - \frac{\delta}{2}\Big)^{\frac{\alpha+1}{\alpha}}\Big]. \end{aligned}$$

Let $n \to \infty$ in the above inequality. Then we have a contradiction:

$$0 \geq \alpha\Big[1 - \Big(1 - \frac{\delta}{2}\Big)^{\frac{\alpha+1}{\alpha}}\Big].$$

Therefore, *Case* (c)-(i) does not occur. Similarly, we can show that *Case* (c)-(ii) does not occur. Let *Case* (c)-(iii) hold. Put $\overline{L} = 1 + \delta\ (\delta > 0)$. Then there is a sequence $\{t_n\}$ satisfying

$$t_1 < t_2 < \cdots < t_n < t_{n+1} < \cdots;\ \lim_{n\to\infty} t_n = \infty;$$
$$w'(t_n) = 0, \quad \text{and} \quad w(t_n) > 1 + (\delta/2), \quad n \in \mathbb{N}.$$

As in the previous Cases, we can get a contradiction.

This completes the proof. □

The following simple lemma is a variant of Gronwall's lemma:

Lemma 3.4: *Let $f, g \in C[t_0, \infty)$, and $c \geq 0$ be a constant such that $f(t), g(t) \geq 0$, and*

$$f(t) \leq c + \int_{t_0}^{t} f(s)\mathrm{d}s + \int_{t_0}^{t} g(s)\mathrm{d}s, \quad t \geq t_0.$$

Then

$$f(t) \leq ce^{t-t_0} + \int_{t_0}^{t} e^{t-s} g(s)\mathrm{d}s, \quad t \geq t_0.$$

Proof: Let us put $H(t) = c + \int_{t_0}^{t} f(s)\mathrm{d}s + \int_{t_0}^{t} g(s)\mathrm{d}s$. Then $f(t) \leq H(t)$ and

$$H'(t) = f(t) + g(t) \leq H(t) + g(t), \qquad t \geq t_0,$$

by the assumption. Therefore,

$$\left(e^{-t}H(t)\right)' \leq e^{-t}g(t), \qquad t \geq t_0,$$

and so an integration on $[t_0, t]$ gives.

$$H(t) \leq ce^{t-t_0} + e^{t} \int_{t_0}^{t} e^{-s} g(s)\mathrm{d}s, \qquad t \geq t_0.$$

Since $f(t) \leq H(t)$, the desired estimate of $f(t)$ holds. This completes the proof. □

Now we are in a position to prove Proposition 3.1.

Proof of Proposition 3.1: We may suppose that $u(t) > 0$ and $u'(t) > 0$. Let $w = (u'/u)^{\alpha}$ as in the Proof of Lemma 3.3. We know that $\lim_{t\to\infty} w(t) = 1$. Further put $z(t) = w(t) - 1$. Then $\lim_{t\to\infty} z(t) = 0$ and $z(t)$ satisfies the equation

$$z' = \alpha\big(1 + b(t)\big) - \alpha\big(1 + z\big)^{(\alpha+1)/\alpha}. \tag{3.3}$$

Since

$$\big(1 + x\big)^{\frac{\alpha+1}{\alpha}} = 1 + \frac{\alpha + 1}{\alpha}x + \varphi(x), \quad x > -1 \tag{3.4}$$

for some continuous function φ with $\varphi(x) = O(x^2),\ x \to 0$, equation (3.3) can be rewritten as follows:

$$z' + \beta z = -\alpha\varphi(z) + \alpha b(t),\ \beta = 1 + \alpha\ (> 0).$$

This is equivalent to

$$\big(e^{\beta t}z\big)' = -\alpha e^{\beta t}\varphi(z) + \alpha e^{\beta t}b(t). \tag{3.5}$$

Let us estimate $z(t)$. Since $\lim_{t\to\infty} z(t) = 0$ and $\lim_{x\to 0}\varphi(x)/x = 0$, there is a sufficiently large $T > 0$ satisfying

$$\alpha\big|\varphi\big(z(s)\big)\big| \le |z(s)| \quad \text{for } s \ge T.$$

An integration of both the sides of equation (3.5) on $[T,t]$ gives

$$e^{\beta t}z(t) = c_1 - \alpha\int_T^t e^{\beta s}\varphi(z(s))\mathrm{d}s + \alpha\int_T^t e^{\beta s}b(s)\mathrm{d}s, \tag{3.6}$$

where $c_1 = e^{\beta T}z(T)$. Therefore,

$$e^{\beta t}|z(t)| \le |c_1| + \int_T^t e^{\beta s}|z(s)|\mathrm{d}s + \alpha\int_T^t e^{\beta s}|b(s)|\mathrm{d}s.$$

By Lemma 3.4 we have for $t \ge T$,

$$e^{\beta t}|z(t)| \le |c_1|e^{t-T} + \alpha\int_T^t e^{t-s}\cdot e^{\beta s}|b(s)|\mathrm{d}s,$$

that is,

$$|z(t)| \le c_2 e^{-\alpha t} + \alpha e^{-\alpha t}\int_T^t e^{\alpha s}|b(s)|\mathrm{d}s \tag{3.7}$$

with some constant $c_2 > 0$.

Recall that,

$$\frac{u'(t)}{u(t)} = w(t)^{1/\alpha} = \big(1 + z(t)\big)^{1/\alpha}. \tag{3.8}$$

Since

$$(1+x)^{1/\alpha} = 1 + \frac{1}{\alpha}x + \rho(x), \quad x > -1, \tag{3.9}$$

for some continuous function ρ satisfying $\lim_{x\to 0}\rho(x)/x = 0$, we obtain from (3.8)

$$\begin{aligned}\int_T^t \frac{u'(s)}{u(s)}\mathrm{d}s &= \int_T^t \big(1+z(s)\big)^{1/\alpha}\mathrm{d}s\\ &= \int_T^t \Big[1 + \frac{1}{\alpha}z(s) + \rho\big(z(s)\big)\Big]\mathrm{d}s,\end{aligned}$$

and so

$$e^{-t}u(t) = u(T)\exp\Big(-T + \frac{1}{\alpha}\int_T^t z(s)\mathrm{d}s + \int_T^t \rho(z(s))\mathrm{d}s\Big).$$

To see $u(t) \sim ce^t$ for some constant $c > 0$, it is sufficient to show that $\int^\infty |z(s)|\mathrm{d}s < \infty$ and $\int^\infty |\rho(z(s))|\mathrm{d}s < \infty$. In the following we will show these facts. By (3.7) and (1.4) we find that,

$$\int_T^\infty |z(t)|\mathrm{d}t \le c_2 \int_T^\infty e^{-\alpha t}\mathrm{d}t + \alpha \int_T^\infty e^{-\alpha t} \int_T^t e^{\alpha s}|b(s)|\mathrm{d}s\mathrm{d}t$$
$$\le \text{const} + \int_T^\infty |b(t)|\mathrm{d}t < \infty.$$

Since we may assume that T is sufficiently large, we find from the property of ρ that,

$$\big|\rho\big(z(t)\big)\big| \le |z(t)| \quad \text{for } t \ge T.$$

Therefore,

$$\int_T^\infty \big|\rho(z(t))\big|\mathrm{d}t \le \int_T^\infty |z(t)|\mathrm{d}t < \infty.$$

This completes the proof of Proposition 3.1. □

3.2 Asymptotic form of nontrivial decreasing solutions

We study asymptotic forms of nontrivial solutions $u(t)$ of equation (1.1) satisfying property (ii) of Lemma 2.2: $|u'(t)|,\ |u(t)| \downarrow 0$ as $t \to \infty$. The argument is, in some sense, parallel to that in Section 3.1.

Proposition 3.5: *Every nontrivial solution u of* (1.1) *satisfying property* (ii) *of Lemma 2.2 has the asymptotic form*

$$u(t) \sim ce^{-t} \quad \textit{as } t \to \infty,$$

for some constant $c \neq 0$.

Lemma 3.6: *Let u be a positive solution of* (1.1) *satisfying property* (ii) *of Lemma 2.2, and put $w = (-u'/u)^\alpha$ for sufficiently large t. Then w satisfies the generalised Riccati equation*

$$w' = \alpha w^{(\alpha+1)/\alpha} - \alpha\big(1 + b(t)\big).$$

Lemma 3.7: *Let u be a positive solution of* (1.1) *satisfying property* (ii) *of Lemma 2.2. Then* $\lim_{t\to\infty}\big[-u'(t)/u(t)\big] = 1$.

Proof: Put $p(t) = \big(1 + b(t)\big)^{\alpha/(\alpha+1)}$. Then $\lim_{t\to\infty} p(t) = 1$, and the function $w = (-u'/u)^\alpha$ satisfies

$$w' = \alpha\Big(w^{(\alpha+1)/\alpha} - p(t)^{(\alpha+1)/\alpha}\Big) \tag{3.10}$$

by Lemma 3.6. It suffices to show that $\lim_{t\to\infty} w(t) = 1$. We consider the following three exclusive cases separately:

Case (a): $w(t) \geq p(t)$ near $+\infty$;

Case (b): $w(t) \leq p(t)$ near $+\infty$;

Case (c): $w(t) - p(t)$ changes the sign in any neighbourhood of $+\infty$.

Let *Case* (a) occur. Since $w'(t) \geq 0$ by (3.10), there is a limit $\lim_{t\to\infty} w(t) \equiv L \in [1, \infty]$. Suppose that $L = +\infty$. Since $(\alpha+1)/\alpha > 1$ and $\lim_{t\to\infty} p(t) = 1$, we find from (3.10) that there is a sufficiently large T satisfying

$$w'(t) \geq \frac{\alpha}{2} w(t)^{\lambda} > 0, \quad t \geq T, \quad \lambda = (\alpha+1)/\alpha > 1.$$

So, $w'(t)w(t)^{-\lambda} \geq \alpha/2$, that is

$$\Big(\frac{w(t)^{1-\lambda}}{1-\lambda}\Big)' \geq \frac{\alpha}{2}, \quad t \geq T.$$

Integrating on $[T, t]$, we obtain

$$\frac{w(T)^{1-\lambda}}{\lambda - 1} \geq \frac{\alpha}{2}(t - T), \quad t \geq T.$$

This is an obvious contradiction. Thus $L \in [1, \infty)$. Letting $t \to \infty$ in (3.10), we get $\lim_{t\to\infty} w'(t) = \alpha\big(L^{\frac{(\alpha+1)}{\alpha}} - 1\big)$. Then as pointed out before, we have $L = 1$ as desired.

Case (b) can be treated similarly; and so we find that $\lim_{t\to\infty} w(t) = 1$.

Finally let *Case* (c) occur. Put $\underline{L} = \liminf_{t\to\infty} w(t)$ and $\overline{L} = \limsup_{t\to\infty} w(t)$. Note that $w'(t) > 0$ [resp. $w'(t) < 0$] if and only if $w(t) > p(t)$ [resp. $w(t) < p(t)$].

To prove $\lim_{t\to\infty} w(t) = 1$, we suppose the contrary that this is not the case.

If $\underline{L} = \overline{L} \in [0, \infty)$, then we can show $\underline{L} = \overline{L} = 1$ easily. So we may assume that $\underline{L} < \overline{L}$. We find from (3.10) that $0 \leq \underline{L} \leq 1 \leq \overline{L} \leq +\infty$ (and $\underline{L} < \overline{L}$). There are three possibilities:

Case (c)-(i): $0 \leq \underline{L} < 1 < \overline{L} \leq +\infty$;

Case (c)-(ii): $0 \leq \underline{L} < 1 = \overline{L}$;

Case (c)-(iii): $\underline{L} = 1 < \overline{L} \leq +\infty$.

Let *Case* (c)-(i) hold. Put $\underline{L} = 1 - \delta$ $(0 < \delta \leq 1)$. Then, as in the Proof of Lemma 3.3, we get a sequence $\{t_n\}$ satisfying

$$t_1 < t_2 < \cdots < t_n < t_{n+1} < \cdots; \quad \lim_{n\to\infty} t_n = \infty;$$
$$w'(t_n) = 0 \quad \text{and} \quad w(t_n) < 1 - \frac{\delta}{2} \quad \text{for } n \in \mathbb{N}.$$

Putting $t = t_n$ in (3.10), and letting $n \to \infty$ in the resulting equation, we get

$$0 \leq \alpha\Big[\Big(1 - \frac{\delta}{2}\Big)^{(\alpha+1)/\alpha} - 1\Big].$$

This is an obvious contradiction. Similarly we can show that *Case* (c)-(ii) does not occur. Let (c)-(iii) hold. Put $\overline{L} = 1 + \delta$ $(\delta > 0)$ if $\overline{L} < \infty$. Then as before we can find a sequence $\{t_n\}$ satisfying

$$t_1 < t_2 < \cdots < t_n < t_{n+1} < \cdots; \quad \lim_{n\to\infty} t_n = \infty;$$
$$w'(t_n) = 0 \quad \text{and} \quad w(t_n) > 1 + \frac{\delta}{2} \quad \text{for } n \in \mathbb{N}.$$

As before we can get a contradiction. The case where $\overline{L} = \infty$ can be treated similarly. This completes the proof. □

We are now in a position to prove Proposition 3.5.

Proof of Proposition 3.5. We may assume that $u(t) > 0$ and $u'(t) < 0$. Let $w = (-u'/u)^\alpha$ as in the proof of Lemma 3.7, in which we have proved $\lim_{t\to\infty} w(t) = 1$. Put $z(t) = w(t) - 1$. Then $\lim_{t\to\infty} z(t) = 0$, and $z(t)$ satisfies the equation

$$z' = \alpha(1+z)^{(\alpha+1)/\alpha} - \alpha\big(1 + b(t)\big).$$

By (3.4) we can rewrite this equation into

$$z' - \beta z = \alpha\varphi(z) - \alpha b(t), \quad \beta = 1 + \alpha,$$

where $\varphi(x) = O(x^2)$ as $x \to 0$. It follows that

$$\big(e^{-\beta t} z\big)' = \alpha e^{-\beta t}\varphi(z) - \alpha e^{-\beta t} b(t),$$

and an integration on $[t, \infty)$ gives

$$e^{-\beta t} z(t) = -\alpha \int_t^\infty e^{-\beta s}\varphi\big(z(s)\big)\mathrm{d}s + \alpha \int_t^\infty e^{-\beta s} b(s)\mathrm{d}s. \tag{3.11}$$

As in the proof of Proposition 3.1, for arbitrary number $\varepsilon > 0$ we can find a sufficiently large number $T = T_\varepsilon > 0$ such that

$$\big|\varphi\big(z(s)\big)\big| \le \varepsilon |z(s)|, \quad s \ge T.$$

So, from (3.11) we obtain

$$e^{-\beta t}|z(t)| \le \alpha\varepsilon \int_t^\infty e^{-\beta s}|z(s)|\mathrm{d}s + \alpha \int_t^\infty e^{-\beta s}|b(s)|\mathrm{d}s, \quad t \ge T. \tag{3.12}$$

Let us denote by $I(t)$ the right-hand side of equation (3.12). Then

$$e^{-\beta t}|z(t)| \le I(t), \text{ and } I(t) = o(e^{-\beta t}) \text{ as } t \to \infty. \tag{3.13}$$

Since

$$\begin{aligned} -I'(t) &= \alpha\varepsilon e^{-\beta t}|z(t)| + \alpha e^{-\beta t}|b(t)| \\ &\le \alpha\varepsilon I(t) + \alpha e^{-\beta t}|b(t)|, \end{aligned}$$

we obtain

$$\big(e^{\alpha\varepsilon t} I(t)\big)' \geq -\alpha e^{-(\beta-\alpha\varepsilon)t}|b(t)|. \tag{3.14}$$

From now on we fix $\varepsilon > 0$ so small that $\beta - \alpha\varepsilon > 0$. Then by (3.13) $\lim_{t\to\infty} e^{\alpha\varepsilon t} I(t) = 0$. Therefore an integration of equation (3.14) on $[t, \infty)$ gives

$$0 \leq I(t) \leq \alpha e^{-\alpha\varepsilon t} \int_t^\infty e^{-(\beta-\alpha\varepsilon)s}|b(s)|\mathrm{d}s.$$

By this estimate and the first inequality of equation (3.13) we find that

$$|z(t)| \leq \alpha e^{(\beta-\alpha\varepsilon)t} \int_t^\infty e^{-(\beta-\alpha\varepsilon)s}|b(s)|\mathrm{d}s. \tag{3.15}$$

Since $-u'(t)/u(t) = \big(1 + z(t)\big)^{1/\alpha}$, by (3.9) we have

$$-\frac{u'(t)}{u(t)} = 1 + \frac{1}{\alpha} z(t) + \rho\big(z(t)\big), \tag{3.16}$$

where $\lim_{x\to 0} \rho(x)/x = 0$. Integrating (3.16) on $[T, t]$, we obtain

$$\log \frac{u(T)}{u(t)} = t - T + \frac{1}{\alpha} \int_T^t z(s)\mathrm{d}s + \int_T^t \rho\big(z(s)\big)\mathrm{d}s. \tag{3.17}$$

By (3.15) we find that

$$\begin{aligned}\int_T^\infty |z(s)|\mathrm{d}s &\leq \alpha \int_T^\infty e^{(\beta-\alpha\varepsilon)s}\Big(\int_s^\infty e^{-(\beta-\alpha\varepsilon)r}|b(r)|dr\Big)\mathrm{d}s \\ &\leq c_1 \int_T^\infty |b(s)|\mathrm{d}s < \infty\end{aligned}$$

for some constant $c_1 > 0$. Similarly, since $\rho(x) = o(x)$ as $x \to 0$, we find that

$$\int_T^\infty |\rho\big(z(s)\big)|\mathrm{d}s \leq c_2 \int_T^\infty |z(s)|\mathrm{d}s < \infty$$

for some constant $c_2 > 0$. Therefore (3.17) implies that

$$\log \frac{u(T)}{u(t)} = t + c_3 + o(1) \text{ as } t \to \infty$$

for some constant $c_3 \in \mathbb{R}$, and so

$$u(t) \sim ce^{-t} \quad \text{as } t \to \infty,$$

for some constant $c > 0$.

This completes the proof of Proposition 3.5. □

As stated before, it is found that Theorem 1.1 is a direct consequence of Propositions 3.1 and 3.5.

References

Bodine, S. and Lutz, D.A. (2015) *Asymptotic Integration of Differential and Difference Equations*, Lecture Notes in Mathematics 2129, Springer, Switzerland.

Coppel, W.A. (1965) *Stability and Asymptotic Behavior of Differential Equations*, Heath, Boston.

Došlý, O. and Řehák, P. (2005) *Half-linear Differential Equations*, Elsevier, Amsterdam.

Došlý, O. and Řezíčková, J. (2011) 'An asymptotic formula for solutions of nonoscillatory half-linear differential equations', *Arch. Math. (Brno)*, Vol. 47, pp.69–75.

Hartman, P. (1982) *Ordinary Differential Equations*, Birkhäuser, Boston.

Luey, S. and Usami, H. (2021) 'Asymptotic forms of solutions of perturbed half-linear ordinary differential equations', *Arch. Math. (Brno)*, Vol. 57, pp.27–39.

Pátíková, Z. (2008) 'Asymptotic formulas for non-oscillatory solutions of perturbed half-linear Euler equation', *Nonlinear Anal.*, Vol. 69, pp.3281–3290.