Master Thesis

Study of Teaching Material Concerning Conics

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APPROVAL

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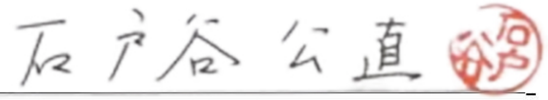

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PREFACE

The main purpose of this thesis to focus on **study of teaching material concerning conics** for developing Cambodian textbooks which concerns how to draw the conics by using computer graphic (BASIC programming) for making teaching material and writing my own exercises in high school. Therefore, I have studied deeply on the conics, especially focused on the pencil of conics to classify the conics, find the rules and generalization, solved the problems, comparing analytic method and synthetic method, generalization, specialization and analogy to find out the new similar problems, drawing graphs to be better (properties preserve by projection), made problems for students and solving it.

These concepts of this thesis are necessary need to study conics. Any way, the curriculum of high school mathematics of my country contain a unit of conics and since my country put great deal of effort to developing science and technologies, the Ministry of Education of our country attaches importance for teaching conics. For this purpose, high school teachers must teach thier students about how to get the formulas and solve system of linear equations. However, there are few teachers who have enough knowledge about conics. Even teachers in higher education know neither the fundamental theory nor enough real example of conics. During I have taught in my country, I have seen many books which publisher have never thought about how to draw conics in the correct way and made some interesting teaching material for making students like studying mathematics. Most of the students and teachers always use only hand to draw some conics by hand. So they tend to draw the picture in the wrong way easily and tend to make the students confuse which curve is represented. So I should prepare my thesis as the following content.

In chapter 1, we concern about all of the posible basic idea of generalization, specialization, analogy, discovery by analogy, analogy and induction and making general problems of any geomatric figures to cover this chapter. And I applied these idea to investigate the high school excercises for guessing the similar excercises by using basic idea and conics to solve it in easier and shorter way.

In chapter 2, we focus on conics. The students can identify locus of point on plan, the equation of the tangent line of a circle, the equation of parabola, the equation of tangent of parabola, the equation of normal line of parabola, the angle between tangent line and parabola and apply it into daily life. Identify equation of ellipse, eccentricity, relation between circle and ellpise, coordinate polar of ellipse, tangent line of an ellipse, angle between tangent line and ellipse. Identify equation of hyperbola, assymptotes, tangent line of hyperbola, angle between tangent line and hyperbola and generalization.

In chapter 3, we concern about tranformation, the students and teachers can know about the relation of transformations, isometries, similarity tranformation, linear tranformation, affine transformation, projective transformation and discovering the theorem by using affine transformation, projective projections concern on how to draw tangent lines and using projective tranformation how to make scale, draw rotation, rotational symmetry, draw reflections, lines and points of symmetry, translation using coordinate, translation by repeat reflections and generalization.

In chapter 4, we focus on pencil of conics. The students can know deeply about the definition of lines pass through two points, how to draw conics by using pencil of lines, pencil of circles, pencil of conics, classification of conics and intersection of two conics. The students can know the world of conics and how to use conics for daily life, especially, know the importance of conics in the world.

In chapter 5, we focus on making and solving problems for covering all chapters above to make the students know deeply about the sample of problems and solve it by using comparison of analytic method and synthetic method to solve the problems, making and solving problems by using analogy of conics in generallization.

However, this thesis can not cover all the importance of the world of conics to the real problems or natural sciences, not to mention the current development of conics, irrespective of the endeavers to provide a comprehensive study of the world of conics. I am strongly expect that, this thises can help the students and teachers more colorful of conic in my country. I am looking for watching from all readers carefully in order to find my faults.

Contents

Chapter 1

Generalization, Specialization and Analogy

1.1 Introduction

In this chapter, we quote from the book of Polya [P,Chapter II].
In this chapter, observation is very importain. We try to use these ideas to observe some properties of conics, in order to find out their similar problems. It will illustrates quite correctly the role of generalization, specialization, and analogy in inductive reasoning to examine less meager, more colorful illustrations and, before that, discuss generalization, specialization, and analogy, these great sources of discovery, for their own sake.

1.2 Generalization

According to [P], generalization is passing from the consideration of a given set of objects to that of a larger set, containing the given one. $\cdots\cdots$ We often generalize in passing from just one object to a whole class containing that object. The section contains examples as follows:
Example1: We pass from the consideration of triangles which have three sides to polygons with n sides where n $\geq$ 3.
Example2: Another example, we consider the trigonometric functions of the acute angles to arbitrary angles α. This example we change the restriction of angle $0 \leq \alpha \leq \pi/2$ to general angle.
Example3: Formula of tangent line to conic
Using matrix, the equation of parabola $y^2 = 4py$ can be written

$$\begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -2p \\ 0 & 1 & 0 \\ -2p & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0,$$

The equation of ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ can be written

$$\begin{bmatrix} x & y & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{a^2} & 0 & 0 \\ 0 & \frac{1}{b^2} & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}=0, \text{ and}$$

the equation of hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ can be written

$$\begin{bmatrix} x & y & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{a^2} & 0 & 0 \\ 0 & -\frac{1}{b^2} & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}=0,$$

They can be generalized as follow

${}^tXAX=0$

where A is symmentric matrix and $X=\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$

Similarly, the equation of tangent line to the parabola $y^2=4px$, ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ and hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ at point $P_0=(x_0,y_0)$ are

$$y_0y=2p(x+x_0)\Longleftrightarrow \begin{bmatrix} x_0 & y_0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & -2p \\ 0 & 1 & 0 \\ -2p & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}=0,$$

$$\frac{x_0x}{a^2}+\frac{y_0y}{b^2}=1\Longleftrightarrow \begin{bmatrix} x_0 & y_0 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{a^2} & 0 & 0 \\ 0 & \frac{1}{b^2} & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}=0,$$

$$\frac{x_0x}{a^2}-\frac{y_0y}{b^2}=1\Longleftrightarrow \begin{bmatrix} x_0 & y_0 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{a^2} & 0 & -2p \\ 0 & -\frac{1}{b^2} & 0 \\ -2p & 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}=0,$$

respectively. In matrix form, they can be generalized as follow: The tangent line to the conic ${}^tXAX=0$ at $P_0=(x_0,y_0)$ is

${}^tX_0AX=0$, where $X_0=\begin{bmatrix} x_0 \\ y_0 \\ 1 \end{bmatrix}$

1.3 Specialization

Specialization is passing from the consideration of a given set of objects to that of a smaller set, contained in the given one. Very often we specialize in passing from a whole class of object to just one object contained in the class.
Example1: We pass from the polygons which have n sides to the regular polygones of n equal sides so that the regular polygon is a specialization of the polygon.
Example2: In the ellipse, we see that when the semi major axis and the semi minor axis equal, then it became the circle, so that circle is a specialization of ellipse.
Example3: If we want to check some assertions about prime numbers, we pick up some prime numbers, say 13, and we examine whether that general assertion is true or false for just this prime 13.

1.4 Analogy

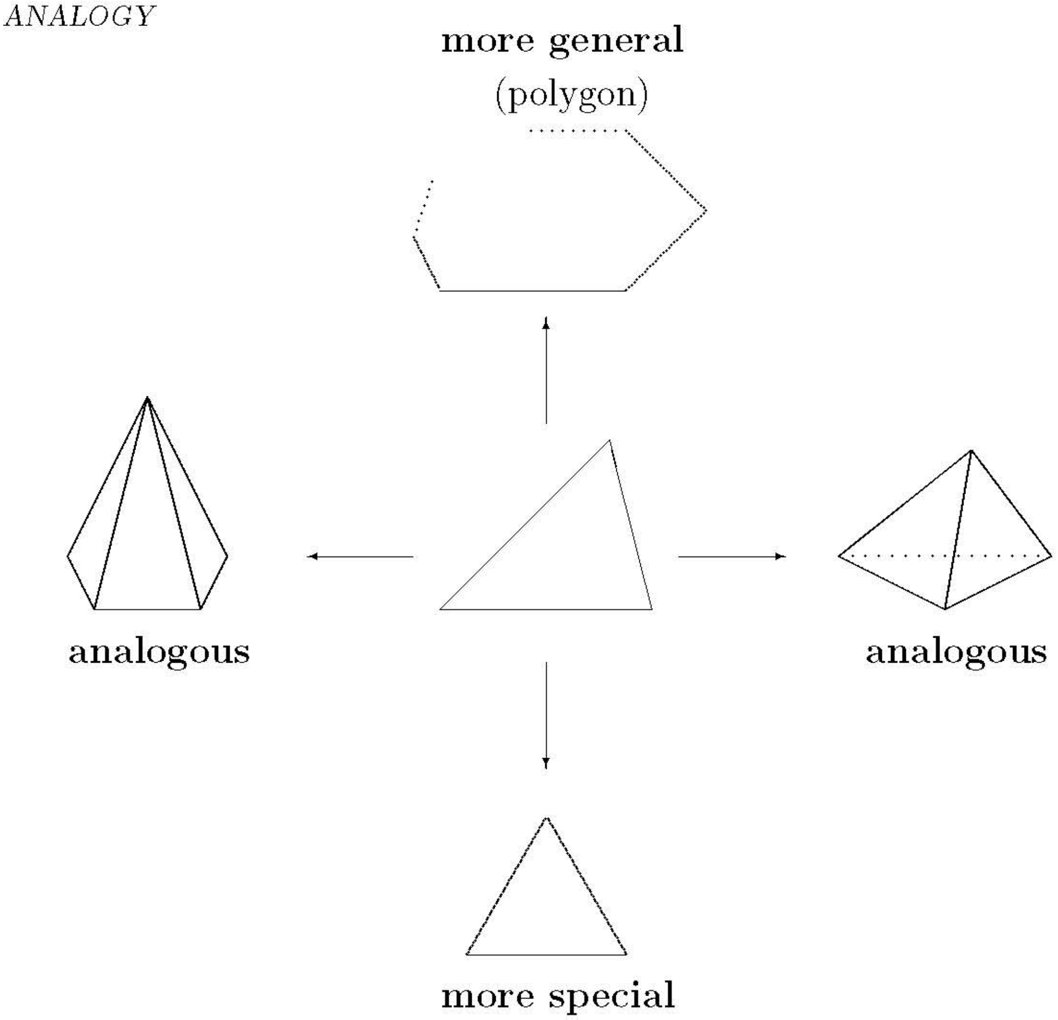
Analogy is a sort of similarity on more definite and more conceptual levels. There is nothing vague or questionable in the concepts of generalization and specialization. Yet as we start discussing analogy we tread on a less solid ground. Similar objects agree with each other in some aspect. Two systems are analogous, if they agree in clearly definable relations of their respective parts.
Example1: A triangle in a plane is analogous to a tetrahedron in space. Because in the plane, 2 straight lines can not include a finite figure but 3 stright lines may include a triangle. In space, 3 planes can not include a finite figure but 4 planes may include a tetrahedron. The relation of the triangle to the plane is the same as that of the terahedron to the space in so far as both the triangle and the tetrahedron are bounded by the minimume number of simple bounding elements. Hence, it is an analogy.
Example2: We may reguard a triangle and a pyramid as analogous figure. On the one hand ,take a segment of a straight line, and on the other hand a polygon. Connect all points of the segment with a point outside the line of the segment, and we obtained triangle. Conect all points of the polygon with a point outside the plane of the polygon, and we obtain a pyramid. In the same manner, we may regard a parallelogram and a prism as analogous figures. In fact, move a segment or a polygon parallel to itself,across of its line or plane, and the one will describe a parallelogram, the other a prism.
Example3: The ellipsoid is analogous to the ellipse because there are many

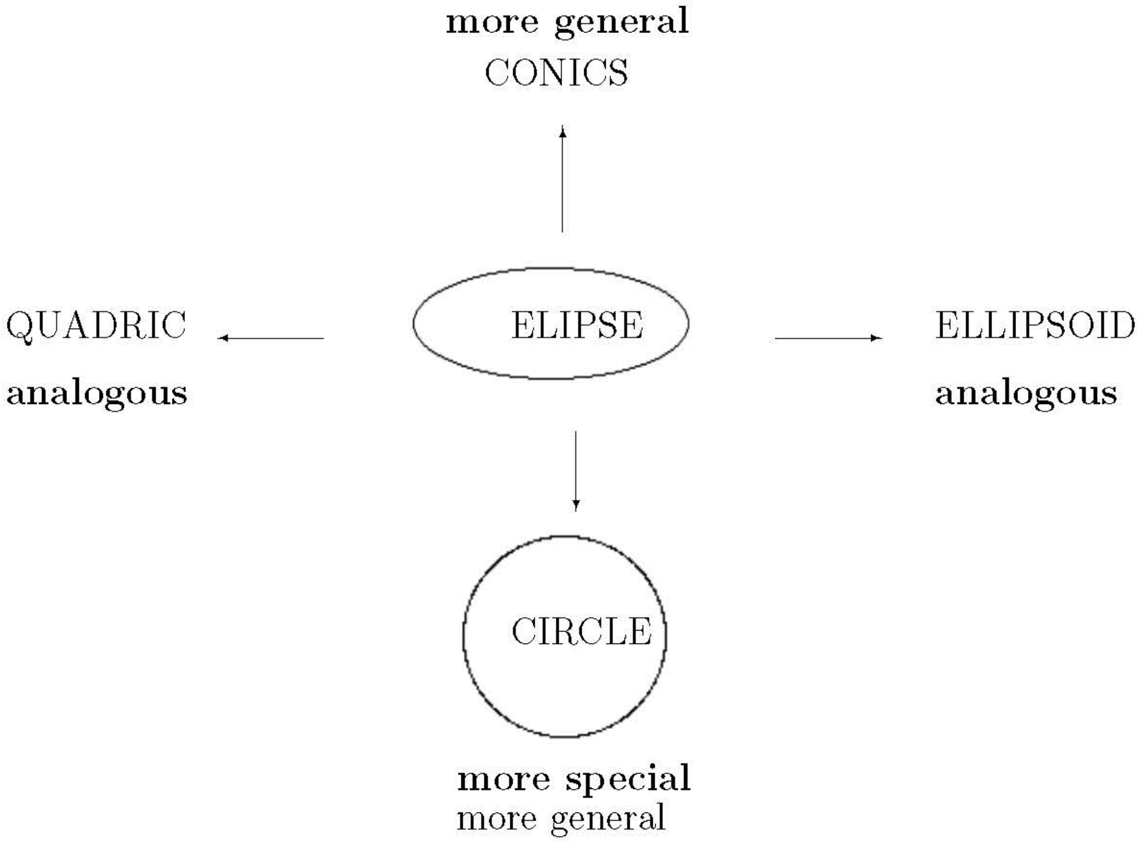
ellipses in the ellipsoid.

Example4: Let point $P(p, q)$ on the conic ${}^t\tilde{\mathrm{X}}A\tilde{\mathrm{X}}= 0 \cdots (\star)$ then we obtain ${}^t\tilde{\mathrm{P}}A\tilde{\mathrm{P}}= 0$

where $\tilde{\mathrm{P}}= \begin{bmatrix} p \\ q \\ 1 \end{bmatrix}$. Let line $\ell : ux + vy + c = 0$ be a tangent line to this conics curve. By the above section, we obtain ${}^t\tilde{\mathrm{U}}A^{-1}\tilde{\mathrm{U}}= 0 \cdots (\star\star)$, where $\tilde{\mathrm{U}}= \begin{bmatrix} u \\ v \\ 1 \end{bmatrix}$.

Then, we say that $(\star)$ and $(\star\star)$ are analogous.


more general
(polygon)
analogous
analogous
more special


more general
CONICS
QUADRIC
ELIPSE
ELLIPSOID
analogous
analogous
CIRCLE
more special
more general

1.5 Discovery by Analogy

Analogy seems to have a share in all discoveries, but in some it has the lion's share. we wish to illustrate this by an example which is not quite elementary, but is of historic interest and far more impressive than any quite elementary example of which we can think.

(1) Consider a few elementary algebric facts essential to Euler's discovery.If the equation of dergree n

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = 0$$

has n different roots

$$\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n$$

the polynormial on its left hand side can be represented as a product of n linear factors,

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^2 = a_n(x - \alpha_1)(x - \alpha_2)\cdots(x - \alpha_n).$$

By comparing the terms with the same power of x on both sides of this identity, we derive the well known relations between the roots and the coeficients of an equation, the simplest of which is

$$a_{n-1} = -a_n(\alpha_1 + \alpha_2 + \cdots + \alpha_n),$$

we find this by comparing the terms with x^{n-1}.

There is another way of presenting the decomposition in linear factors. If none of the root $\alpha_1, \alpha_2, \cdots, \alpha_n$ is equal to 0, or (which is the same) if a_0 is different from 0, we have also

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = a_0(1 - \frac{x}{\alpha_1})(1 - \frac{x}{\alpha_2})\cdots(1 - \frac{x}{\alpha_n})$$

and

$$a_1 = -a_0(\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \cdots + \frac{1}{\alpha_n}).$$

There is still another variant. Suppose that the equation is of degree $2n$, has the form

$$b_0 - b_1x^2 + b_2x^4 - \cdots + (-1)^n b_n x^{2n} = 0$$

and $2n$ diffirent roots

$$\beta_1, -\beta_1, \beta_2, -\beta_2, \cdots, \beta_n, -\beta_n.$$

$$b_0 - b_1x^2 + b_2x^4 - \cdots + (-1)^n b_n x^{2n}$$
$$= b_0(1 - \frac{x^2}{\beta_1^2})(1 - \frac{x^2}{\beta_2^2}) \cdots (1 - \frac{x^2}{\beta_n^2})$$

and

$$b_1 = b_0(\frac{1}{\beta_1^2} + \frac{1}{\beta_2^2} + \cdots + \frac{1}{\beta_n^2}).$$

(2) Consider about Euler's equation

$$\sin x = 0$$

or

$$\frac{x}{1} - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \frac{x^7}{1.2.3 \cdot 7} + \cdots = 0.$$

The left hand side has an infinity of terms, is of "infinitive degree". Therefore, it is no wonder, says Euler, that there is an infinity of roots

$$0, \pi, -\pi, 2\pi, -2\pi, 3\pi, -3\pi, \cdots.$$

Euler discards the root 0. He devides the left hand side of the equation by x, the linear factor corresponding to the root 0, and obtains so the equation

$$1 - \frac{x^2}{2.3} + \frac{x^4}{2.3.4.5} - \frac{x^6}{2.3.4.5.6.7} + \cdots = 0$$

with the roots

$$\pi, -\pi, 2\pi, -2\pi, -2\pi, 3\pi, -3\pi, \cdots$$

We have seen an analogous situation before, under (1), as we discussed the last variant of the decomposition in linear factors. Euler concludes, by analogy, that

$$\frac{\sin x}{x} = 1 - \frac{x^2}{2.3} + \frac{x^4}{2.3.4.5} - \frac{x^6}{2.3.4.5.6.7} + \cdots$$
$$= (1 - \frac{x^2}{\pi^2})(1 - \frac{x^2}{4\pi^2})(1 - \frac{x^2}{9\pi^2}) \cdots,$$
$$\frac{1}{2.3} = \frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \cdots,$$
$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}.$$

This is the series that withstood the efforts of Jacques Bernoulli- but it was a daring conclusion.

(3) Euler knew very well that his conclusion was daring. " The method was new and never used yet for such a purpose ", he wrote 10 years later. He saw some objections himself and many objections were raised by his mathematical friends when they recovered from their first admiring surprise.
Yet Euler had his reason to trust his discovery. First of all, the numerical value for the sum of the series which he has computed before, agreed to the last place with $\pi^6/6$. Comparing further coefficients in his expression of $\sin x$ as a product, he found the sum of other remarkable series, as that of the reciprocals of the fourth powers,

$$1+\frac{1}{16}+\frac{1}{81}+\frac{1}{256}+\frac{1}{625}+\cdots=\frac{\pi^4}{90^o}$$

Again, he examined the numerical value and agian he found agreement.
(4) Euler also tested his method on other example. Doing so he succed in rederiving the sum $\pi/6$ for Jacques Bernoullis series by various modifications of his first approach. He succeed also in rediscovering by his method the sum of an importain series due Leibnitz.

Let us discuss the last point. Let us consider, following Euler, the equation

$$1-\sin x=0$$

It has the roots

$$\frac{\pi}{2},-\frac{3\pi}{2},\frac{5\pi}{2},-\frac{7\pi}{2}-\frac{9\pi}{2},-\frac{11\pi}{2},\ldots$$

Each of these roots is, however, a double root. (The curve $y=\sin x$) does not intersect the line $y=1$ at these abscissas, but is tangent to it. The derivative of the left hand side vanishes for the same values of x, but not the second derivatives.) Therefore, the equation

$$1-\frac{x}{1}+\frac{x^3}{1.2.3}-\frac{x^5}{1.2.3.4.5}+\cdots=0$$

has the roots

$$\frac{\pi}{2},\frac{\pi}{2},-\frac{3\pi}{2},-\frac{3\pi}{2},\frac{5\pi}{2},\frac{5\pi}{2},-\frac{7\pi}{2},-\frac{7\pi}{2},\ldots$$

and Euler's analogical conclusion leads to the decomposition in linear factors

$$\begin{aligned}1-\sin x&=1-\frac{x}{1}+\frac{x^3}{1.2.3}-\frac{x^5}{1.2.3.4.5}+\cdots\\&=(1-\frac{2x}{\pi})^2(1+\frac{2x}{3\pi})^2(1-\frac{2x}{5\pi})^2(1+\frac{2x}{7\pi})^2\cdots\end{aligned}$$

Comparing the coeffient of x on both sides, we obtain

$$-1 = -\frac{4}{\pi} + \frac{4}{3\pi} - \frac{5}{5\pi} + \frac{4}{7\pi} \cdots,$$
$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots$$

This is Leibnitz's celebrated series, Euler's daring procedure led to a known result. For our method which may appear to some as not reliable enough, a great confirmation comes here to light. Therefore, we should not doubt at all of the other things which are derived by the same method.
(5) We continued the numerical verifications discribed above under (3), examined more series and more decimal places, and found agreement in all cases examined. We tried other approaches, too, and, finally, we succeeded in verifying not only numerically, but exactly, the value $\pi^2/6$ for Jacques Bernoulli's series. We found a new proof. This proof, although hidden and ingenious, was based on more usual considerations and was accepted as completely was satifactorily verified.
These arguments, it seems, convienced Eluer that his result was correct.

1.6 Analogy and Induction

In this section, we wish to learn something about the nature of inventive and inductive reasoning. What can we learn foregoing story?
In strict logic, it was an outright fallacy: Euler's decisive step was daring, he applied a rule to a case for which the rule was not made, a rule about algebric equations to an equation which is not justified. Yet it was justified by analogy, by the analogy of the most successful achievements of rising science that we call "Analysis of the Infinite." Other mathematicians, before Euler, passed from finite differences to infinitely small differences, from sums with a finite number of terms to sums with an infinity of terms, from finite products to infinite products. And so Euler passed from equation of finite degree (algebric equations) to equations of infinite degree, applying the rules made for the finite to the infinite.
This analogy, this passage from the finite to the infinite, is beset with pitfalls.In this case how can we avoid these? Euler was a genius, some people will answer, and of couse that is no explanation at all. He had shrewd reason for trusting his dicovery. We can understand his reasons with a little common sense, without any miraculous insight specific to genuis.

Euler's reasons for trusing his discovery, summarized in the foregoing, are not demonstrative. Euler does not reexamine the grounds for his conjucture, for his daring passage from the finite to the infinite; he examines only its consequences. He requards the verification of any such consequence as an argument in favor of his conjuctur. He accepts both approximative and exact verifications, but seems to attach more weight to the latter. He examines also the consequence of closely related analogous conjucture and he requard the verification of such a consequence as an arqument for his conjucture.
Euler's reasons are, in fact, inductive. It is a typical inductive procedure to examine the consequences of a conjucture and to judge it is on the basis of such an examination. In scientific research as in ordinary life, we believe, or ought to believe, a conjecture more or less according as in observable consequences agree more or less with the facts.
In short, Euler seems to thinks. He seems to accept certain principles: *A conjucture becomes more cridible by the verification of any new consequence.*And ***A conjecture becomes more cridible if an analogous conjucture becomes more credible.***
Are the principles underlying the process of induction of this kind?
Example: 1 Consider the conjecture E. We requard the equation

$$\sin x = x(1-\frac{x^2}{\pi^2})(1-\frac{x^2}{4\pi^2})(1-\frac{x^2}{9\pi^2})\cdots$$

as a conjucture; we call it the "conjucture E". Follow Euler, we wish to investigate this conjucture inductively. Derive from E, we know that $\sin(-x) = -\sin x$. Does this fact agree with E?
Answer:

$$\begin{aligned}\sin(-x) &= -x(1-\frac{(-x)^2}{\pi^2})(1-\frac{(-x)^2}{4\pi^2})(1-\frac{(-x)^2}{9\pi^2})\cdots \\ &= -x(1-\frac{x^2}{\pi^2})(1-\frac{x^2}{4\pi^2})(1-\frac{x^2}{9\pi^2})\cdots \\ &= \sin x\end{aligned}$$

Therefore this equality a gree with E.
Example2: Pridict from E and verify the value of the infinite product

$$(1-\frac{1}{4})(1-\frac{1}{9})(1-\frac{1}{16})\cdots(1-\frac{1}{n^2})\cdots.$$

Answer From **E**, we obtain

substitute

$$x=\pi, then\ \sin\pi = \pi(1-\frac{\pi^2}{\pi^2})(1-\frac{\pi^2}{4\pi^2})(1-\frac{\pi^2}{9\pi^2})\cdots(1-\frac{\pi^2}{n^2\pi^2})$$
$$= \pi(1-\frac{1}{4})(1-\frac{1}{9})(1-\frac{1}{16})\cdots(1-\frac{1}{n^2})$$

Devide E by $1-\frac{x}{\pi}$

$$\frac{\sin x}{1-\frac{x}{\pi}} = x(1-\frac{x}{\pi^2})(1-\frac{x^2}{4\pi^2})(1-\frac{x^2}{9\pi^2})\cdots(1-\frac{x^2}{n^2\pi^2})$$

When $x \longrightarrow \pi$, RHS $\longrightarrow \pi(1+1)(1-\frac{1}{4})(1-\frac{1}{9}\cdots(1-\frac{x^2}{n^2\pi^2})\cdots$
Let $x-\pi=u$, LHS $=\frac{\pi\sin(u+\pi)}{-u}=\frac{-\pi\sin u}{-u}=\frac{\pi\sin u}{u}$
When $u\longrightarrow 0$, then LHS $\longrightarrow \pi$
We obtain $\pi=\pi(1+1)(1-\frac{1}{4})(1-\frac{1}{9})\cdots(1-\frac{1}{n^2})$
Therefore,

$$(1-\frac{1}{4})(1-\frac{1}{9})\cdots(1-\frac{1}{n^2})\cdots=\frac{1}{2}\cdots\cdots(\star)$$

This is pridicture.
Proof $(\star)$

$$1-\frac{1}{n^2}=\frac{n^2-1}{n^2}=\frac{(n-1)(n+1)}{n^2}=\frac{(n-1)}{n}.\frac{(n+1)}{n}$$
$$1-\frac{1}{2^2}=\frac{1}{2}.\frac{3}{2}$$
$$1-\frac{1}{3^2}=\frac{2}{3}.\frac{4}{3}$$
$$1-\frac{1}{4^2}=\frac{3}{4}.\frac{5}{4}$$
$$(1-\frac{1}{4})(1-\frac{1}{9})\cdots(1-\frac{1}{n^2})=(\frac{1}{2}.\frac{3}{2})(\frac{2}{3}.\frac{4}{3})(\frac{3}{4}.\frac{5}{4})\cdots(\frac{n-2}{n-1}.\frac{n}{n-1})(\frac{n-1}{n}.\frac{n+1}{n})$$
$$=\frac{1}{2}.\frac{n+1}{n}$$

When $n\longrightarrow\infty$ then $\frac{1}{2}.\frac{n+1}{n}\longrightarrow\frac{1}{2}$
Therefore, this predicture is true.

1.7 General Problems:

Problem 1. We have a circle with radus r and center $O(0,0)$. We draw two tengent lines ℓ_1, ℓ_2 which perpendicular to the x-axis at point A and B respectively. Let T be any points on circle. We draw a tangent line of circle at T an intersect at ℓ_1, ℓ_2 at point P and Q respectively. Prove that $PT : TQ = AP : BQ$.

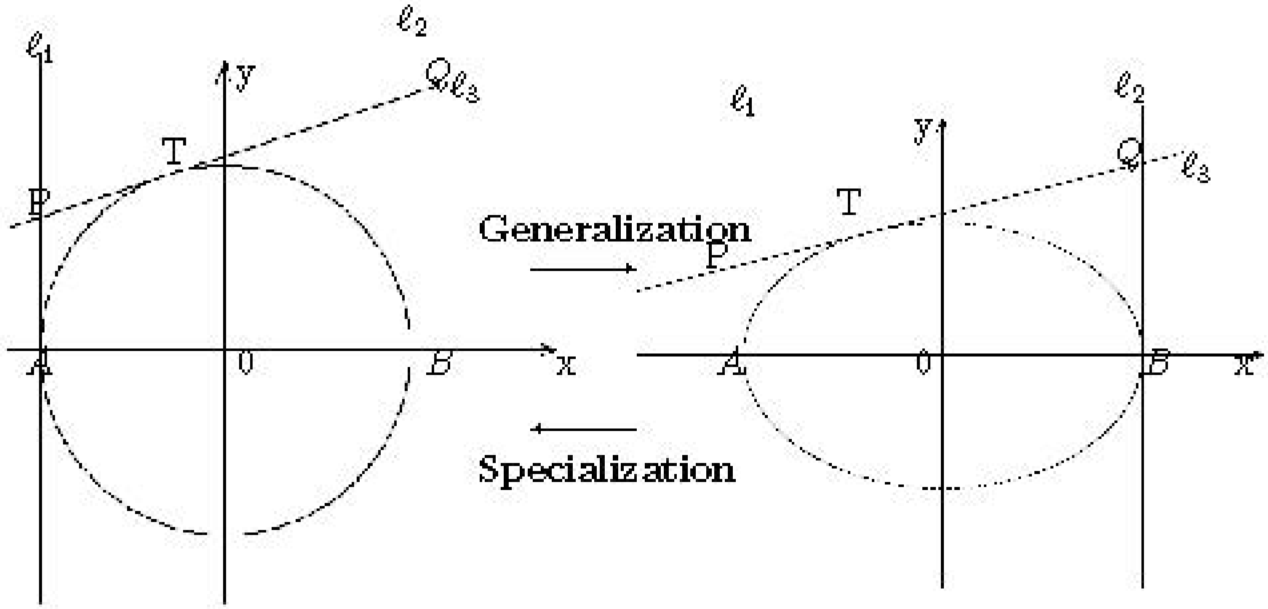


Fig (a)

Fig (b)

Proof:
Since ℓ_1 and ℓ_3 are the tangent line of circle and $P = (\ell_1) \cap (\ell_2)$, then $AP = PT$. Similarly, Q is an intersection of the tangent line ℓ_2 and ℓ_3, then $BQ = QT$. Therefore, $PT : TQ = AP : BQ$.

We know that circle is the specialization of ellipse. So we will guess that the problem of ellipse in Fig(b) is analogy with the problem in Fig(a),that is $PT : TQ = AP : BQ$. It is true, so we are going to prove as the following:
Let $T = (x_1, y_1)$ then $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2} = 1$ and the tangent line at T is $\ell_3 : \frac{x_1 x}{a^2}+\frac{y_1 y}{b^2} = 1$.

Since $(\ell_1) \cap (\ell_2) = P$ and $\ell_1 : x = -a$, we have

$$\frac{-ax_1}{a^2} + \frac{y_1 y}{b^2} = 1$$

$$y = \frac{b^2}{y_1}(1 + \frac{x_1}{a})$$

$$= \frac{b^2}{ay_1}(a + x_1)$$

Hence, $P = \left(-a, \frac{b^2}{ay_1}(a + x_1)\right)$, so we obtain

$$AP = \left|\frac{b^2}{ay_1}(a + x_1)\right|$$

$$PT = \sqrt{(x_1 + a)^2 + \left(y - \frac{b^2}{ay_1}(a + x_1)\right)^2}$$

$$= \frac{1}{|ay_1|}\sqrt{b^2(a^2 - x_1^2)(a + x_1)^2 + b^2(a - \frac{x_1^2}{a}) - b^2(a + x_1)}$$

$$= \frac{b}{|ay_1|}\sqrt{(a^2 - x_1^2)(x_1 + a)^2 + \frac{b^2}{a^2}x_1^2(x_1 + a)^2}$$

$$= \left|\frac{b^2}{a^2 y_1}(a + x_1)\right| \sqrt{x^4 + (b^2 - a^2)x_1^2}$$

$$= \frac{AP}{|a|}\sqrt{a^4 + (b^2 - a^2)x_1^2}$$

Since $\ell_2 : x = a$ intersects the tangent line ℓ_3 at Q, then

$$x_Q = (a, y_Q) = \frac{b^2}{ay_1}(a - x_1)$$

Hence, $BQ = \left|\frac{b^2}{ay_1}(a - x_1)\right|$ and similarly calculation, we obtain

$$TQ = \frac{BQ}{|a|}\sqrt{a^4 + (b^2 - a^2)x_1^2}$$

Therefore, $PT : TQ = AP : BQ$. So the specialization problem and the generalization problem are very importain to find tha analogy problem. Therefore, the editor should imaging the special case and find it generalization then they can quess the problem which are analogous to each other.

Problem 2. Given an ellipse $E : \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ passes through x-axis at A and B. We draw a line $\ell : x = x_1$ intersects E at $P = (x_1, y_1)$ and $P' = (x_1, -y_1)$. Let $R = (x, y)$ be an intersection point of the two lines AP and $P'B$. If we translate line ℓ from A and B, find the locus of point R.

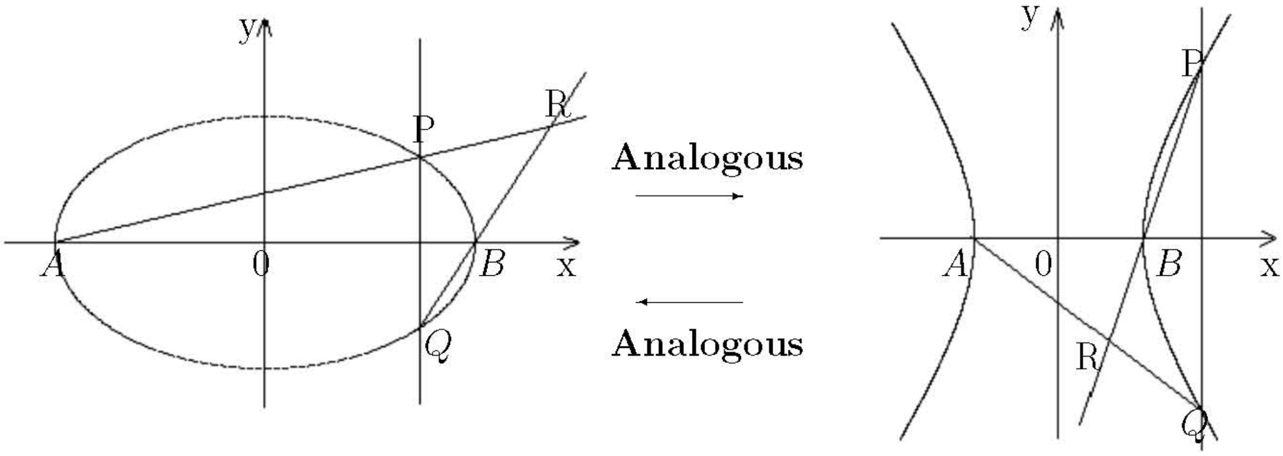


Fig (a) **Fig (b)**

Proof.
The intersection point of line $\ell : x = x_1$ and ellipse E are defined by

$$\frac{x_1^2}{a^2} + \frac{y^2}{b^2} = 1, \ then\ y = \pm\frac{b}{a}\sqrt{a^2 - x_1^2}.$$

Then $P = (x_1, b\sqrt{1 - \frac{x_1^2}{a^2}})$ and $Q = (x_1, -b\sqrt{1 - \frac{x_1^2}{a^2}})$.

Hence the equation of lines

$$AP : y = \frac{b\sqrt{1-\frac{x_1^2}{a^2}}}{a+x_1}(x-x_1) + b\sqrt{1-\frac{x_1^2}{a^2}}$$

$$QB : y = \frac{b\sqrt{1-\frac{x_1^2}{a^2}}}{a-x_1}(x-x_1) - b\sqrt{1-\frac{x_1^2}{a^2}}$$

Since R is an intersection point between AP and BQ, then we can find as the follow.

$$\frac{b\sqrt{1-\frac{x_1^2}{a^2}}}{a+x_1}(x-x_1) + b\sqrt{1-\frac{x_1^2}{a^2}} = \frac{b\sqrt{1-\frac{x_1^2}{a^2}}}{a-x_1}(x-x_1) - b\sqrt{1-\frac{x_1^2}{a^2}}$$

$$b\sqrt{1-\frac{x_1^2}{a^2}}\left(\frac{1}{a-x_1} - \frac{1}{a+x_2}\right)(x-x_1) = 2b\sqrt{1-\frac{x_1^2}{a^2}}$$

$$\frac{x_1}{a^2-x_1^2}(x-x_1) = 1$$

$$x = \frac{a^2-x_1^2}{x_1} + x_1 = \frac{a^2}{x_1}$$

By substituting $x = \frac{a^2}{x_1}$ into equation AP, we obtain

$$y = b\sqrt{1-\frac{x_1^2}{a^2}}\left\{\frac{a^2-x_1^2}{ax_1+x_1^2} + 1\right\}$$

$$= b\frac{a}{x_1}\sqrt{1-\frac{x_1^2}{a^2}} = \frac{bx}{a}\sqrt{1-\frac{x_1^2}{a^2}}$$

$$\frac{y}{b} = \frac{x}{a}\sqrt{1-\frac{x_1^2}{a^2}}$$

By square both sides we obtain

$$\frac{y^2}{b^2} = \frac{x^2}{a^2}\left(1-\frac{x_1^2}{a^2}\right) = \frac{x^2}{a^2}\left(1-\frac{a^2}{x^2}\right)$$

Therefore, the locus of R is a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

If we compare the graphs of Fig(a) and Fig(b), we well suspect that they are analogy. Hence, the locus of R in Fig(b) may be an ellipse, i.e., if we have hyperbola $H : \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ then the locus of R as mension in Fig(b) is an ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Is it true?

So that we shall prove the problem in Fig(b) as the following.
We have point $A = (-a, 0)$ and $B = (a, 0)$ and line $\ell : x = x = 1$ intersect hyperbola H at

$$P = \left(x_1, b\sqrt{\frac{x_1^2}{a^2} - 1}\right) \text{ and } Q = \left(x_1, -b\sqrt{\frac{x_1^2}{a^2} - 1}\right).$$

Then, we obtain equation of lines

$$AP : y = \frac{b\sqrt{\dfrac{x_1^2}{a^2} - 1}}{a + x_1}(x - x_1) + b\sqrt{\frac{x_1^2}{a^2} - 1}$$

$$QB : y = \frac{b\sqrt{\dfrac{x_1^2}{a^2} - 1}}{a - x_1}(x - x_1) - b\sqrt{\frac{x_1^2}{a^2} - 1}$$

Hence, the intersection point of these lines can be found as the follows.

$$\frac{b\sqrt{\dfrac{x_1^2}{a^2} - 1}}{a + x_1}(x - x_1) + b\sqrt{\frac{x_1^2}{a^2} - 1} = \frac{b\sqrt{\dfrac{x_1^2}{a^2} - 1}}{a - x_1}(x - x_1) - b\sqrt{\frac{x_1^2}{a^2} - 1}$$

$$b\sqrt{\frac{x_1^2}{a^2} - 1}\left(\frac{1}{a - x_1} - \frac{1}{a + x_1}\right)(x - x_1) = 2b\sqrt{\frac{x_1^2}{a^2} - 1}$$

$$\frac{x_1}{a^2 - x_1^2}(x - x_1) = 1$$

$$x = \frac{a^2 - x^2}{x_1} + x_1 = \frac{a^2}{x_1}$$

Substitute $x = \dfrac{a^2}{x_1}$ into the equation of line AP, we have

$$y = b\sqrt{\frac{x_1^2}{a^2} - 1}\left\{\frac{a^2 - x_1^2}{ax_1 + x_1^2} + 1\right\}$$

$$= b\frac{a}{x_1}\sqrt{\frac{x_1^2}{a^2} - 1} = \frac{bx}{a}\sqrt{\frac{x_1^2}{a^2} - 1}$$

$$\frac{y}{b} = \frac{x}{a}\sqrt{\frac{x_1^2}{a^2} - 1} = \frac{x}{a}\sqrt{\frac{a^2}{x^2} - 1}$$

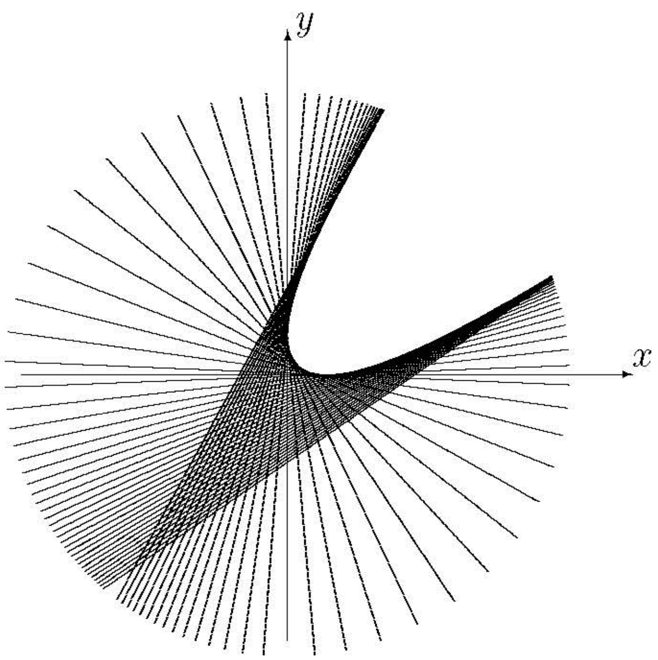
Hence, we square both sides we abtain

$$\frac{y^2}{b^2} = \frac{x^2}{a^2}\left(\frac{a^2}{x^2} - 1\right) = 1 - \frac{x^2}{a^2}$$

Therefore, the locus of R is an ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Thus, our assertion is true. So analoguos is very powerful to predict the similar problem.

Problem 3. Given two points $A(0, s)$ and $B(t, 0)$. If point A, B change and $s + t = 1$, then find the shape of curve which bounded by line pass through points A, B.

Proof.



Let ℓ_t be line pass through points A and B. Then the equation of line ℓ_t is

$$\frac{x}{t} + \frac{y}{s} = 1, \quad or \quad sx + ty - st = 0.$$

Since $s + t = 1$ then s=1-t. Hence $\ell_t : (1 - t)x + ty = (1 - t)t$, and

$$\ell_{(t+\Delta t)} : (1 - (t + \Delta t))x + (t + \Delta t)y = [1 - (t + \Delta t)](t + \Delta t)$$

In order to find the shap of curve which bounded by lines ℓ_t, we have to find the intersection point of ℓ_t and $\ell_{(t+\Delta t)}$ as $\Delta t \longrightarrow 0$.
From line ℓ_t we have

$$y = \frac{t-1}{t}x + (1 - t)$$

substituting into $\ell_{(t+\Delta t)}$, we have

$(1-(t+\Delta t))x+(t+\Delta t)\left(\frac{t-1}{t}x+(1-t)\right)=[1-(t+\Delta t)](t+\Delta t)$

$\left\{1-(t+\Delta t)+(t+\Delta t)\frac{t-1}{t}\right\}x=[1-(t+\Delta t)](t+\Delta t)+(t+\Delta t)(t-1)$

$\{t-(t+\Delta t)t+(t+\Delta t)(t-1)\}\,x=(t+\Delta t)t\,\{[1\Delta-(t+\Delta t)]+(t-1)\}$

$\{t-(t+\Delta t)\}x=(t+\Delta t)t(-\Delta t)$

$x=t(t+\Delta t)$

Then

$$y=(t-1)(t+\Delta t)+(1-t)=(t-1)(t+\Delta t-1)$$

When $\Delta t\longrightarrow 0$ then the locus of bounded curve is denoted by the parametric equation

$$P_t:\begin{cases} x= & t^2 \\ y= & (t-1)^2 \end{cases}$$

Then $t=\pm\sqrt{x}$ and $y=t^2-2t+1$ or

$$\begin{aligned} y&=x\mp 2\sqrt{x}+1\\ y-x-1&=\mp 2\sqrt{x}\\ (y-x-1)^1&=4x\\ y^2+x^2+1-2xy+2x-2y&=4x\\ x^2+y^2-2xy-2x-2y+1&=0\cdots(\star) \end{aligned}$$

Thus, we obtain matrix $A=\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$, and $\tilde{\mathrm{A}}=\begin{bmatrix}1 & -1 & -1\\ -1 & 1 & -1\\ -1 & -1 & 1\end{bmatrix}$

Now we need to check $det(A)$.

Since $det(A)=1-1=0$, then we have to check $\det(\tilde{\mathrm{A}})$. Hence

$$\det(\tilde{\mathrm{A}})\begin{vmatrix}1 & -1 & -1\\ -1 & 1 & -1\\ -1 & -1 & 1\end{vmatrix}=\begin{vmatrix}1 & -1 & -1\\ 0 & 0 & -2\\ 0 & -2 & 0\end{vmatrix}=4\ \neq 0$$

Then, the quadratic equation $(*)$ is a parabola.

Chapter 2

Conics

2.1 Introduction

In mathematics, a conic section (or just conic) is a curve obtained by intersecting a cone (more precisely, a right circular conical surface) with a plane. In analytic geometry, a conic may be defined as a plane algebraic curve of degree 2. It can be defined as the locus of points whose distances are in a fixed ratio to some points, called a focus, and some lines, called a directrix. The conic sections were named and studied as long ago, when Apollonius of Perga undertook a systematic study of their properties.
The three types of conics are the ellipse, parabola, and hyperbola. The circle can be considered as a fourth type (as it was by Apollonius) or as a kind of ellipse. The circle and the ellipse arise when the intersection of cone and plane is a closed curve. The circle is obtained when the cutting plane is parallel to the plane of the generating circle of the cone for a right cone as in the picture at the top of the page this means that the cutting plane is perpendicular to the symmetry axis of the cone. If the cutting plane is parallel to exactly one generating line of the cone, then the conic is unbounded and is called a parabola. In the remaining case, the figure is a hyperbola. In this case, the plane will intersect both halves (nappes) of the cone, producing two separate unbounded curves.
Various parameters are associated with a conic section:

conic	circle	ellipse	parabola	hyperbola
equation	$x^2+y^2=a^2$	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	$y^2=4ax$	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$
eccentricity(e)	0	$\frac{\sqrt{a^2-b^2}}{a}$	1	$\frac{\sqrt{a^2+b^2}}{a}$
linear eccentricity(c)	0	$\sqrt{a^2-b^2}$	a	$\sqrt{a^2+b^2}$
semi-latus rectum(l)	a	$\frac{b^2}{a}$	2a	$\frac{b^2}{a}$
focal parameter(p)	∞	$\frac{b^2}{\sqrt{a^2-b^2}}$	2a	$\frac{b^2}{\sqrt{a^2+b^2}}$

Conic sections (or just conics) are exactly those curves that, for a point F, a line L not containing F and a non-negative number e, are the locus of points whose distance to F equals e times their distance to L. F is called the focus, L the directrix, and e the eccentricity.

. The linear eccentricity (c) is the distance between the center and the focus (or one of the two foci).
. The latus rectum (2l) is the chord parallel to the directrix and passing through the focus (or one of the two foci).
. The semi-latus rectum (l) is half the latus rectum.
. The focal parameter (p) is the distance from the focus (or one of the two foci) to the directrix.

The following relations hold:
. pe=l
. ac=c.

2.2 Loci and Equations

Example1. Find the locus of points on plane that are equidistant from the sides of angle SOT and lie in the interior of the angle.
Answer. If we take a point P in the interior of the angle $\angle SOT$ so that P is equidistant from the sides OS and OT. If we let the foot of perpendiculars be A and B, then $\triangle OBP$ are congruent.

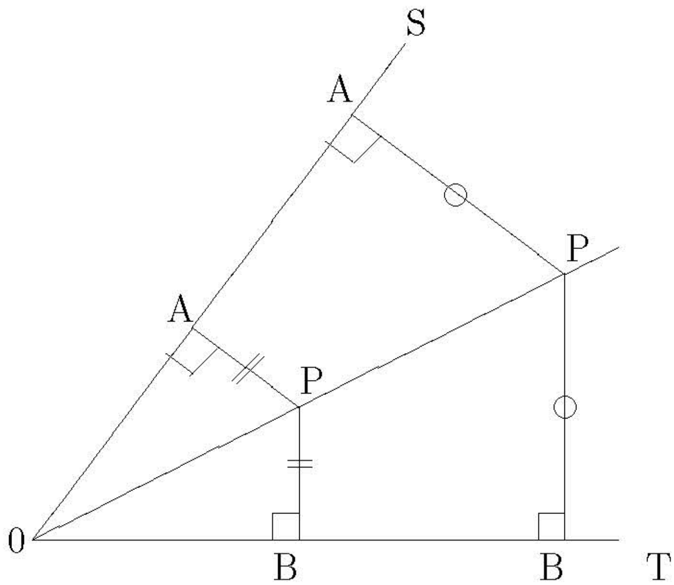
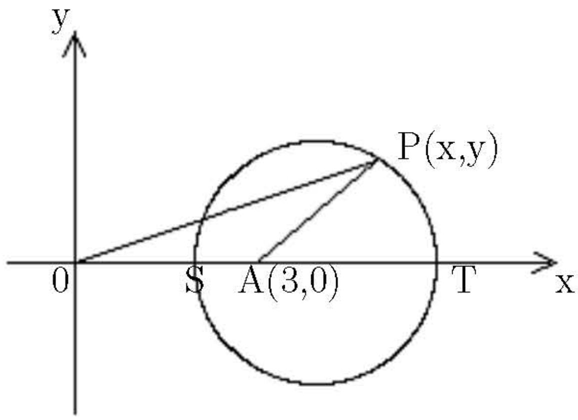


Figure II.2.1 Locos is a bisector

We have $\angle POA = \angle POB$. Therefore the locus of such points is the bisector of $\angle SOT$.

Example2. (Apollonius' Circl



Find the locus of points whose ratio distance from the points $O(0,0)$ a $A(3,0)$ is $OP : AP = 2:1$.

Answer. Let the coordinates of P (x,y). We have $OP^2 = x^2 + y^2$ a $AP^2 = (x-3)^2 + y^2$
By the condition given $2AP = OF$

Figure II.2.2. Locos is a circle with the center $(4,0)$

Hence $4AP^2 = OP^2$ we have

$$4(x-3)^2 + y^2 = x^2 + y^2$$
$$3x^2 + 3y^2 - 24x + 36 = 0$$
$$(x+4)^2 + y^2 = 4$$

Therefore the locus is the circle with the center$(4,0)$ and the radius 2
the points start devide OA internally and externally to the ratio 2 to 1 are $S(2,0)$ and $T(6,0)$. The segment ST is the diameter of the circle

$$(x-4)^2 + y^2 = 4.$$

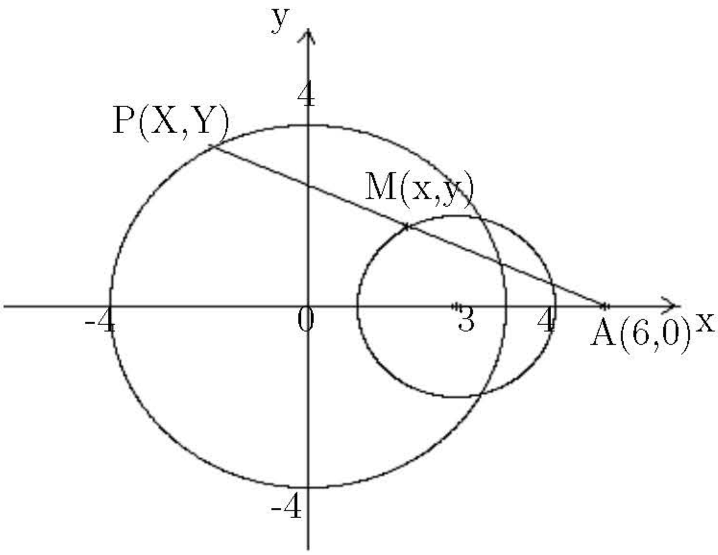
Example3. The point $A(6,0)$ and the circle $x^2 + y^2 = 16$ are given. When a point P moves on the circle, find the locus of the midpoint M of the seqment AP.

Answer. Let the coordinates of P and M be $P(X,Y)$ and $M(x,y)$ respectively
As P is on the circle, we have $X^2 + Y^2 = 16 \cdots$ (i)
As M is the midpoint of AP, we have

$(x, y) = (\frac{X+6}{2}, \frac{Y}{2})$
Therefore, $(X, Y) = (2x - 6, 2y)$.
Thus by (i), we have

$$\begin{aligned}(2x-6)^2 + (2y)^2 &= 16\\ 4x^2 + 4y^2 - 24x + 36 &= 16\\ x^2 + y^2 - 6x + 5 &= 0\\ (x-3)^2 + y^2 &= 4.\end{aligned}$$

Therefore, the locus of M is the circle with the center(3,0) and the radius 2.



Fiqure II.2.3. Locos is a circle with the center (3,0) and radius 2.

2.3 Circles and lines

Example1. Find the equation of a circle which passes through $A(2, 2)$, $B(-4, 0)$ and $C(0, -2)$. And, find the coordinates of its center and the lenght of its radius.

Answer. Suppose that the equation of the circle is $(x - a)^2 + (y - b^2) = r^2$.

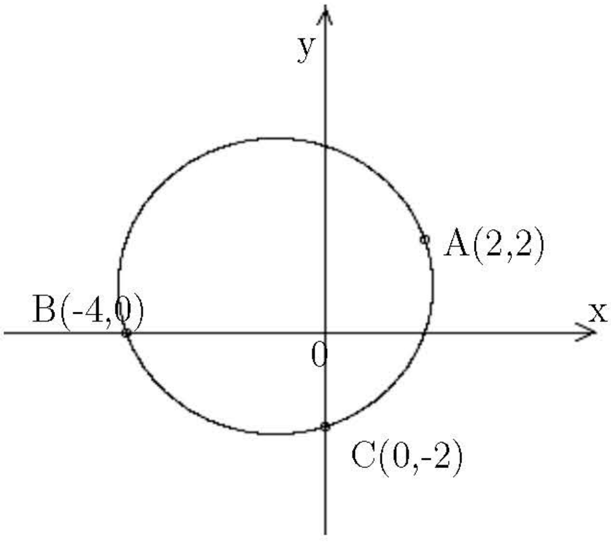


Figure II.3.1. Circle with center is (-1,1),and the radius is $\sqrt{10}$

Then, from the conditions of the equation, we have
$(2 - a)^2 + (2 - b)^2 = r^2 \cdots (i)$,
$(-4 - a)^2 + (0 - b)^2 = r^2 \cdots (ii)$,
$(0 - a)^2 + (-2 - b)^2 = r^2 \cdots (iii)$.
By (i)-(ii), we have 1-a-2b $= 0 \cdots$ (iv).

By (ii)-(iii), we have 3+2a-b $= 0 \cdots$ (v).
Therefore by (iv) and (v), we have a=-1 and b=1.
Thus $r = \sqrt{10}$.
The equation of the circle is $(x+1)^2 + (y-1)^2 = 10$.
The center is (-1,1), and the radius is $\sqrt{10}$.

Example2. Find the number of common points of the circle $x^2 + y^2 = 10$ and the line $y = 2x - 5$. And find the coordinates of the common points.

Answer

$x^2 + y^2 = 10$ (i), $y = 2x - 5$ (ii)
By (i) and (ii), we have

$$\begin{aligned} x^2 + (2x-5)^2 &= 10 \\ 5x^2 - 20x + 15 &= 0 \\ x^2 - 4x + 3 &= 0 \\ (x-1)(x-3) &= 0 \\ x &= 1, 3 \end{aligned}$$

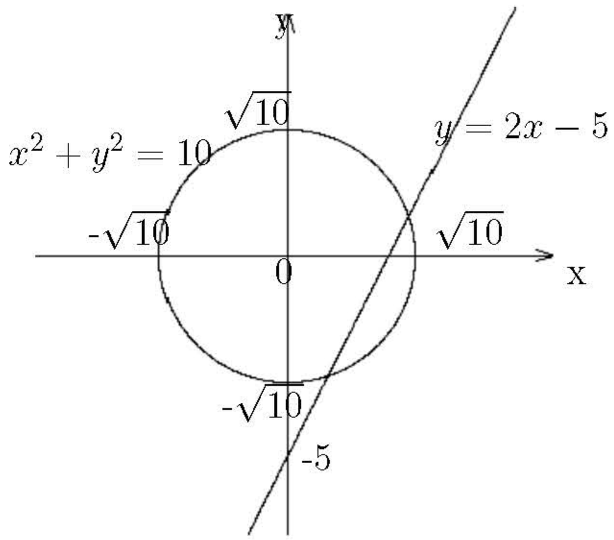


Figure II.3.2a Common points of a circle $x^2 + y^2 = 10$ and a line $y = 2x - 5$

By (ii), we have
$(x, y) = (1, -3), (3, 1)$.
There are two common points.
When a system of equation $(x-a)^2 + (y-b)^2 = r^2$ and $y = px + q$ is given, let D be the discriminant of the equation $(x-a)^2 + \{(px+q) - b\}^2 = r^2$. Then

a) $D > 0 \iff$ Two common points$\cdots$ the line and the circle intersect at two points.
b) $D = 0 \iff$ One common point $\cdots$ The line is tangent to the circle.
c) $D < 0 \iff$ No common point $\cdots$ the line and the circle do not meet.

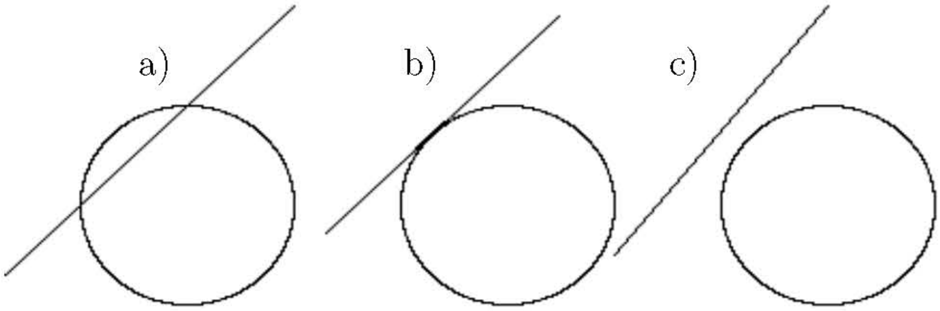


Figure II.3.2b. Common point of a circle and a line

Example3. Determind the rank of k so that a circle $x^2 + y^2 = 5$ and a line $y = 2x + k$ intersect at two p
Answer. $x^2 + y^2 = 5$ (i), $y =$
By (i) and (ii), we have

$$x^2 + (2x + k)^2 = 5$$
$$5x^2 - 4kx + k^2 - 5 = 0 \cdots$$

If the discriminant of $(\star)$ is p
then the circle and a line ha
common points, we have:

$$D = (4k)^2 - 4 \times 5 \times (k^2 -$$
$$= -4k^2 + 100$$
$$= -4(k + 5)(k - 5) > ($$

Therefore $-5 < k < 5$.
Thus $x^2 + y^2 = 5$ and $y = 2x +$
two common pointss,
when $-5 < k < 5$

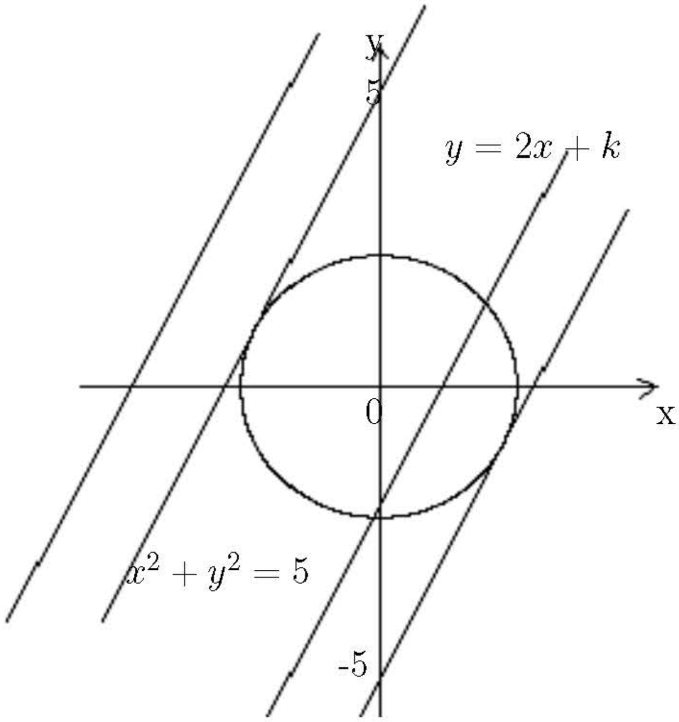


Figure II.3.3. Common point of a circle $x^2 + y^2 = 5$ and aline $y = 2x + k$
Remark:

The equation of the tangent at $P(x_1, y_1)$ on circle $(x - a)^2 + (y - b)^2 = r^2$ is given by $(x_1 - a)(x - a) + (y_1 - b)(y - b) = r^2$

Proof. The tangent at a point (x_1, y_1) is $(x_1 - a)(x - a) + (y_1 - b)(y - b) = r^2$
Diffirentiate both sides of $(x - a)^2 + (y - b)^2 = r^2$ we have

$$2(x - a) + 2y'(y - b) = 0 \Longleftrightarrow y' = -\frac{(x - a)}{(y - b)}.$$

The slope of line ℓ at point $P(x_1, y_1)$ is: $-\dfrac{(x_1 - a)}{(y_1 - b)}$.
When $x_1 \neq a, y_1 \neq b$ equation of tangent line is:
$y - y_1 = -\dfrac{(x_1 - a)}{(y_1 - b)}(x - x_1)$
$(y_1 - b)(y - y_1) = -(x_1 - a)(x - x_1)$
$y_1 y - y_1^2 - by + by_1 + x_1 x - x_1^2 - ax + ax_1 = 0$
$(-y_1^2 + 2by_1 - b^2) + y_1 y - by - by_1 + b^2 + (-x_1^2 + 2ax_1 - a^2) + x_1 x - ax - ax_1 + a^2 = 0$
$-(y_1 - b)^2 - (x_1 - a)^2 + (x_1 - a)(x - a) + (y_1 - b)(y - b) = 0$
$(x_1 - a)(x - a) + (y_1 - b)(y - b) = (y_1 - b^2) + (x_1 - a)^2$
$(x_1 - a)(x - a) + (y_1 - b)(y - b) = r^2 \cdots (\star)$(Because $(y_1 - b)^2 + (x_1 - a)^2 = r^2$).
Therefore $(x_1 - a)(x - a) + (y_1 - b)(y - b) = r^2$
When $x_1 = a$ we have $y_1 = r \pm b$ $((x - a)^2 + (y - b)^2 = r^2)$.
Therefore, the tangent line is: $y = r + b$ or $y = r - b$.
When $y_1 = b$ we have $x_1 = r \pm a \cdots (\star)((x - a)^2 + (y - b)^2 = r^2)$.
Therefore, the tangent line is $x = r + a$ or $x = r - a$.
Both of two cases above satisfy to equation $(\star)$
Therefore eqation of tangent line at point $P(x_1, y_1)$ of circle is:

$$(x_1 - a)(x - a) + (y_1 - b)(y - b) = r^2.$$

2.4 Parabolas

2.4.1 Definition of Parabola

Given a point F and a line ℓ that does not pass through F. The parabola with focus F and directrix ℓ is the locus of points whose the distance from F is equal to their distance from ℓ (i.e $\ell \not\ni F$)

2.4.2 Equation of Parabola

From the definition above (II.4.1), to find the equation of a parabola, draw a line FH from the focus F perpendicular to the directrix ℓ. Then define coordinate axis by taking the midpoint O of FH as the origin, taking the perpendicular bisector of line segment FH as the x-axis, and taking the line FH as the yaxis. For the non-zero p , the coordinates of F are $(0, p)$, and the equation of ℓ is $y = -p$.
Draw line PQ perpendicular to ℓ from point $P(x, y)$ on the locus, and then

$$PQ = |y + p|, PF = \sqrt{x^2 + (y - p)^2}.$$

Therefore, the condition of the locus $PQ = PF$ can be written as

$$|y + p| = \sqrt{x^2 + (y - p)^2}.$$

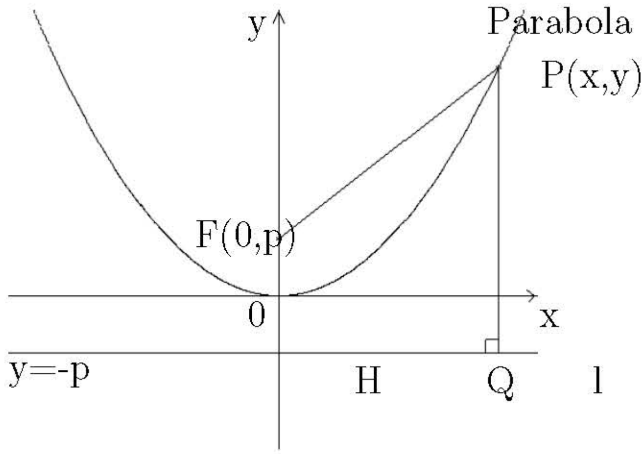
Squaring both sides of this equation, we obtain

$$(y + p)^2 = x^2 + (y - p)^2$$

Rearranging this equation,

$$4py = x^2 \ or \ y = \frac{1}{4p}x^2$$

This is the equation of the parabola, and the y-axis is the axis.



Therefore the equation of a parabola in which the focus is (0,p) and the directrix is y = -p is

$$4py = x^2 \quad or \quad y = \frac{1}{4p}x^2$$

Remark:

For the parabola wich the focus is (0,p) and directrix is y = -p has the equation is

$$\boxed{y = ax^2 \ or \ x^2 = \frac{1}{a}y, \ which \ a = \frac{1}{4p} \Longleftrightarrow p = \frac{1}{4a}}$$

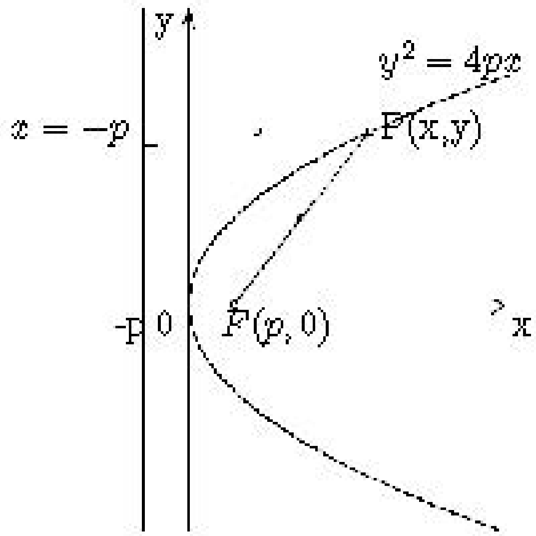
. Coordinate of focus $(0, \frac{1}{4a})$

. Equation of directrix $y = -\frac{1}{4a}$

Example1. A point $F(p, 0)$ and a line $g : x = -p$ (p :constant) are given. Find the locus of point P such that the distances from F to P and from g to P are equal.

Answer.

Let the coordinates of P be $P(x, y)$.
Then $FP^2 = (x-p)^2 + y^2 \cdots$ (i),
$(The\ distance\ from\ g\ to\ P)^2 = (x+p)^2 \cdots$(ii).
From the equation, we have (i)=(ii).
Thus $(x-p)^2+y^2 = (x+p)^2$ and $y^2 = 4px$.
Therefore the locus of point P is $y^2 = 4px$.
This is also the definition of parabola.
For given parabola $y^2 = 4px$, the line $g : x = -p$ is called the directrix and the point $F(p, 0)$ the focus, respectively.



Example2. Find the equations of the following parabolas

1- focus (0,2), dirextrix y = 2

2- focus $(0, \frac{1}{4})$, vertext (0,0)

Answer.

1- By using the equation of parabola $x^2 = 4py$, then $p = 2$
So the equation is $x^2 = 8y$

2- By using the equation of parabola $x^2 = 4py$, then $p = \frac{1}{4}$
So the equation is $x^2 = y$

The curve created by reflecting parabola (1) above with respect to the line y = x is the figure represented by the equation $y^2 = 4px$. This curve is a parabola with the point (p,0)as its focus and the line x = -p as it directrix.

Example3.

Find the equations of the parabolas:

1- focus(-3,0), directrix x = 3

2- focus (1,0), directrix x = -1

Answer.

1- By the equation of the parabola $y^2 = 4px$, then $p = -3$
So the equation is $y^2 = -12x$
2- By the equation of the parabola $y^2 = 4px$, then $p = 1$
So the equation is $y^2 = 4x$

2.4.3 Translating Figure

Let C be a figure represented by the equation f(x,y)=0
i.e (Consider a plane figure as a set of points in R^2)

$$C = \{(x, y)|f(x, y) = 0\}$$

Example: 1
Let C be the bisector line (locus of the points P such that $AP = BP$) of the segment AB, where A:(1,1), B:(0,2).

Answer1.
$C : \{P|AP = BP\}$
$\Longleftrightarrow \{(x, y)|\sqrt{(1-x)^2 + (1-y)^2} = \sqrt{x^2 + (2-y)^2}\}$
Square on the both sides, we obtain
$\Longleftrightarrow \{(x, y)|(x-1)^2 + (y-1)^2 = x^2 + (y-2)^2\}$
$\Longleftrightarrow \{(x, y)|x^2 - 2x + 1 + y^2 - 2y + 1 = x^2 + y^2 - 4y + 4\}$
$\Longleftrightarrow \{(x, y)| - 2x + 2y - 2 = 0\}$
$\Longleftrightarrow \{(x, y)| - x + y - 1 = 0\}$
So the equation of C is given
$x = y - 1$

Another way:
$P = (x, y)$
$P \in C \Longleftrightarrow AP = BP$
$\Longleftrightarrow \{(x, y)|\sqrt{(1-x)^2 + (1-y)^2} = \sqrt{x^2 + (2-y)^2}\}$
Square on the both sides, we obtain
$\Longleftrightarrow \{(x, y)|(x-1)^2 + (y-1)^2 = x^2 + (y-2)^2\}$
$\Longleftrightarrow \{(x, y)|x^2 - 2x + 1 + y^2 - 2y + 1 = x^2 + y^2 - 4y + 4\}$
$\Longleftrightarrow \{(x, y)| - 2x + 2y - 2 = 0\}$
$\Longleftrightarrow \{(x, y)| - x + y - 1 = 0\}$
$\Longleftrightarrow -x + y - 1 = 0$
$\Longleftrightarrow y = x + 1$
So the equation of C is given by $y = x + 1$.

Example: 2

Let C be the locus of points P such that $AP.BP = K$, where $A = (1,0)$, $B = (-1,0)$ and k a positive constant.

Find the equation of C?

Answer.

Let $P = (x,y)$

$P \in C \iff AP.BP = k$

$\sqrt{(x-1)^2+y^2}.\sqrt{(x+1)^2+y^2} = k$

Square on the both sides, we obtain

$\{(x-1)^2+y^2\}\{(x+1)^2+y^2\} = k^2$

$\iff (x^2-2x+1+y^2).(x^2+2x+1+y^2) = k^2$

$\iff (x^2+y^2+1)^2-(2x)^2 = k$

$\iff (x^2+y^2)^2+2(x^2+y^2)+1-4x^2 = k^2$

$\iff (x^2+y^2)^2-2x^2+2y^2 = k-1$

Therefore the equation is $(x^2+y^2)^2-2x^2+2y^2 = 0$.

Choose $P(1,y)$, then we obtain

Let x=1, then

$(1+y^2)^2-2+2y^2 = 0$

$1-2y^2+y^4-2+2y^2 = 0$

Let $X = y^2$ then

$X^2+4X-1 = 0$

$X = -2 \pm \sqrt{5}$

But $X = y^2 \neq 0$, and $-2-\sqrt{5} < 0$

So $X = -2+\sqrt{5}$

$y^2 = -2+\sqrt{5}$

Therefore the equation of C is

$$y = \pm\sqrt{-2+\sqrt{5}}$$

Consider the Polar equation for liminate $(x^2+y^2)^2-2x^2+2y^2 = k^2-1$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

Subsituting to the equation, we obtain

$(r^2)^2-2r^2\cos^2\theta+2r^2\sin^2\theta = k^2-1$

$r^4-2r^2(\cos^2\theta-sin^2\theta) = k^2-1$

$r = 0$ or $r^2-2(\cos^2\theta-\sin^2\theta) = k^2-1$

$r^2-2\cos 2\theta = 0$

$r = \pm\sqrt{2\cos 2\theta}$.
Therefore the equation of C is

$$(2\cos 2\theta)^2 - 4\cos 2\theta \cos^2\theta + 4\cos 2\theta \sin^2\theta = k^2 - 1.$$

2.4.4 Translating Parabola

The equation of the fingure created by the equation $f(x, y) = 0$ by m units along x-axis and n unit along y-axis is given by

$$f(x - m, y - n) = 0.$$

Theorem. We know that
(1) $C : f(x, y) = 0$ $i.e$ $(x, y) \in C \iff f(x, y) = 0$
(2) C' is created from C by translating along vector(m, n)
Let $C : f(x, y) = 0$ and C' be the figure created translating C by m units along x-axis and n unit along y-axis then the equation of C' is given by

$$f(x - m, y - n) = 0.$$

Proof: $\Longrightarrow$

Let (x,y) $\in C'$
C' is created by translating along vector (m, n), then $(x_0, y_0) \in C$ $(\star)$ by translating along vector (m, n)
i.e $(x, y) = (x_0 + m, y_0 + n)$, then $(x_0, y_0) = (x - m, y - n)$ by (1) and $(\star)$, we obtain $f(x_0, y_0) = 0$
Therefore $f(x - m, y - n) = 0$.

Proof: $\Longleftarrow$

Suppose $f(x - m, y - n) = 0$, then by (1), $(x - m, y - n) + (m, n) \in C$ then by(2) $(x - m, y - n) + (m, n) \in C'$
but $(x - m, y - n) = (x, y)$
Therefore $(x, y) \in C'$.

Example
Sketch the parabolas represented by the following equations. Find the coordinates of the focus and the equation of the directrix.
(1) $4y = -x^2 + 4x + 8$

(2) $y^2 - 6y = x$

Answer.

(1) $2y = -x^2 + 4x + 8$

Let $C : 4y = -x^2 + 4x + 8$, then

$$\begin{aligned} 4y &= -(x^2 - 4x - 8) \\ 4y &= -((x-2)^2 - 12) \\ 4y &= -(x-2)^2 + 12 \\ 4(y-3) &= -(x-2)^2 \end{aligned}$$

This is obtained from
$C_0 : 4y = -x^2$ by translating
(x,y)$\longmapsto (x+2, y+3)$

$$\begin{aligned} F_0 &= (0, -1) \\ d_0 : y &= 1 \\ F &= (2, 2) \\ d : y &= 4 \end{aligned}$$

(2) $y^2 - 6y = x$

$$\begin{aligned} y^2 - 6y + 9 &= x + 9 \\ (y-3)^2 &= x + 9 \end{aligned}$$

This is obtain from $y^2 = x$ by translating
(x,y)$\longmapsto (x-9, y+3)$

$$\begin{aligned} y^2 &= x \\ F &= (\frac{1}{4}, 0) \\ d : x &= -\frac{1}{4} \\ F' &= (-\frac{35}{4}, 3) \\ d' : x &= -\frac{37}{4} \end{aligned}$$

2.4.5 Parabola and Stright Lines

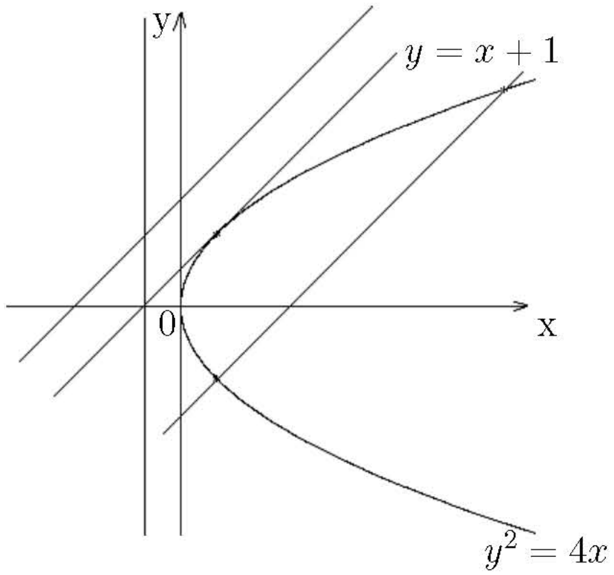
Example. How will the number of common points shared by the parabola $y^2 = 4x$ and the straight line $y = x + k$ change according to the value of k?

Answer. From equation $y^2 = 4x$ and $y = x + k$.
The real solutions of quadratic equation $y^2 - 4y + 4k = 0$ $(\star)$. By eliminating $x = y - k$ into $y^2 = 4x$.
Therefore the number of commons points is equal to the number of real solutions of $(\star)$.
If we take D as the discriminant of equation $(\star)$, then $\frac{D}{4} = 4 - 4k = 4(1 - k)$

(i) For $D > 0$ or $k < 1$, they have two common points.

(ii) For $D = 0$ or $k = 1$, they have one common point.

(iii) For $D < 0$ or $k > 1$, they have no common point.



FigureII.4.5. Common points of parabola $y^2 = 4x$ and line $y = x + k$
If there is one common point, the line and paralola are said to be tangent, and the common point is the tangent point.

2.4.6 Tangent Lines of Parabola

Example. For a parabola $y^2 = 4px$, prove that

1) The equation of its tangent whose slope is m is: $y = mx + \frac{p}{m}$.

2) The equation of the tangent at a point $P(x, y)$ on the parabola is given by: $y = \frac{2p}{y_1}(x + x_1)$.

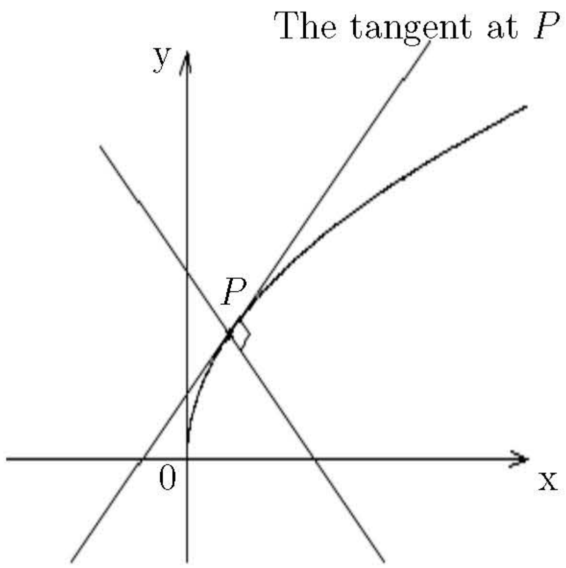
Answer.1) The equation whose the slope is m can be expressed as $y = mx + b$ (i). The intersection points of $y^2 = 4px$ and (i) can be given by the roots of these simultaneous equation. By substituting (i) into $y^2 = 4px$, we obtain $(mx + b)^2 = 4px$

$m^2x^2 + 2(mb - 2p)x + b^2 = 0$.
In order that (i) is tangent to parabola, it is neccesary to be
$(mb - 2p)^2 - m^2b^2 = 4p^2 - 4mbp = 0$. So $b = \dfrac{p}{m}$
Therefore the equation of its tangent whose slope is m is $y = mx + \dfrac{p}{m}$

2) Let the slope of the tangent at P be m and the equation of the tengent be $y - y_1 = m(x - x_1)$ or $y = m(x - x_1) + y_1$ (ii).
Then we have $\{m(x - x_1) + y_1^2\} = 4px$
$m^2x^2 - 2\{m(mx_1 - y_1) + 2p\}x + m(x_1 - y_1)^2 = 0$.
Therefore $x = \dfrac{m(mx_1 - y_1) + 2p \pm \sqrt{0}}{m^2} = \dfrac{m(mx_1 - y_1) + 2p}{m^2}$.
Because this x is equal to x_1, we have$x_1 = \dfrac{m(mx_1 - y_1) + 2p}{m^2}$.
Hence $m = \dfrac{2p}{y_1}$. By (ii), we have $y = \dfrac{2p}{y_1}(x + x_1) + y1$. So $y_1y = 2p(x - x_1) + y_1^2$.
As P is on the parabola $y^2 = 4px_1$, we have $y_1^2 = 4px_1$.
Therefore $yy_1 = 2p(x - x_1) + 4px_1$ and so $y = \dfrac{2p}{y_1}(x + x_1)$.
When $y_1 = 0$, the tangent is $x = 0$ (the y-axis) as $x_1 = 0$.
Thus the equation of the tangent at a point $P(x_1, y_1)$is $y = \dfrac{2p}{y_1}(x + x_1)$.

2.4.7 Equation of Normal Line

The line that is orthogonal to tangent at P on a geometric figure is called the normal line at P.
The slope of the tangent at $p(x_1, y_1)$ on a parabola $y^2 = 4px$ is equal to $\dfrac{2p}{y_1}$.
There for the slope of the normal at P is given by $-\dfrac{y_1}{2p}$, and so the equation of the normal is :
$y - y_1 = -\dfrac{y_1}{2p}(x - x_1)$



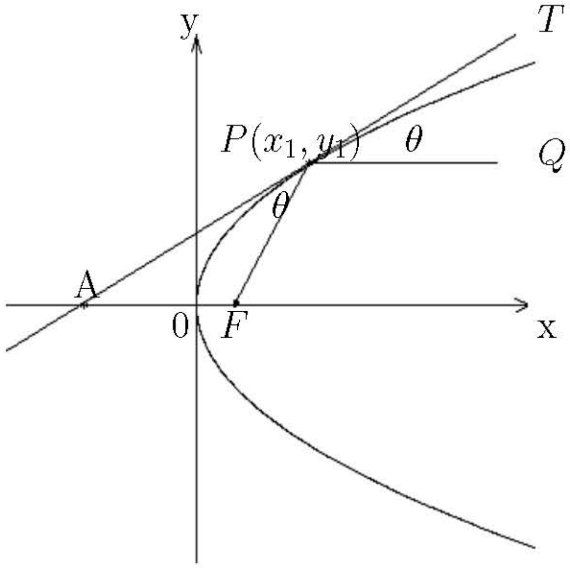
2.4.8 Angle of Tangent Line and Parabola

Example. Let a point $P(x_1, y_1)$ be on a parabola $y^2 = 4px$,(p: positive constant) and a point $F(p, 0)$ be the focus of the parabola. Draw the tangent at the point P on the parabola, and draw a line PQ which is parallel to the x-axis. When the tangent line meets the x-axis at a point A, prove that the angle made by FP and AT is equal to the angle made by PQ and T.

Answer. The equation of the line AT is $yy_1 = 2p(x + x_1)$
Thus, the coordinates of A is $A(-x_1, 0)$
$AF = p - (-x_1) = p + x_1$
On the other hand,

$$\begin{aligned} PF^2 &= (p - x_1)^2 + y_1^2 \\ &= p^2 - 2px_1 + x_1^2 + y_1^2 \\ &= p^2 - 2px_1 + x_1^2 + 4px_1 \\ &= (p + x_1)^2. \end{aligned}$$

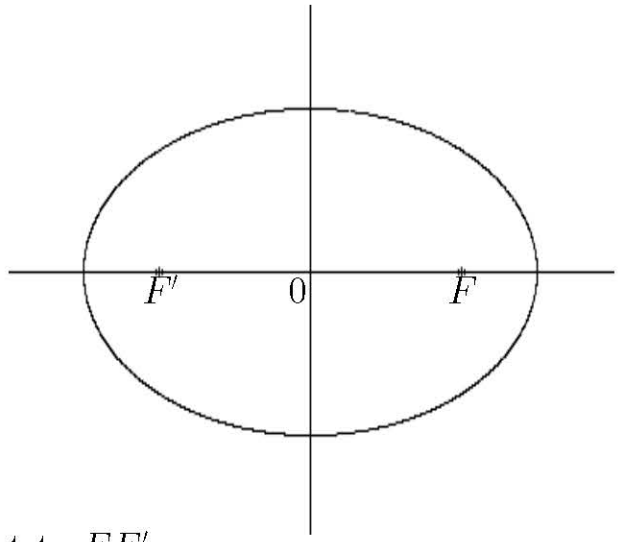
Therefore $PF = p + x_1$.
since $AF = PF$.
Therefore $\triangle APF$ is an isoscales traingle with $AF = PF$.
Hence $\angle FAP = \angle FPA$.
As $PQ//AF$, $\angle FAP = \angle QPT$.
Therefore $\angle FPA = \angle QPT$.



Example above shows the important property of a parabola.
"Rays of light parallel to the axis meet at the focus of the parabola after reflection."
This property of a parabol is used for parabola antennas for satellite broadcasting and headlights of cars.

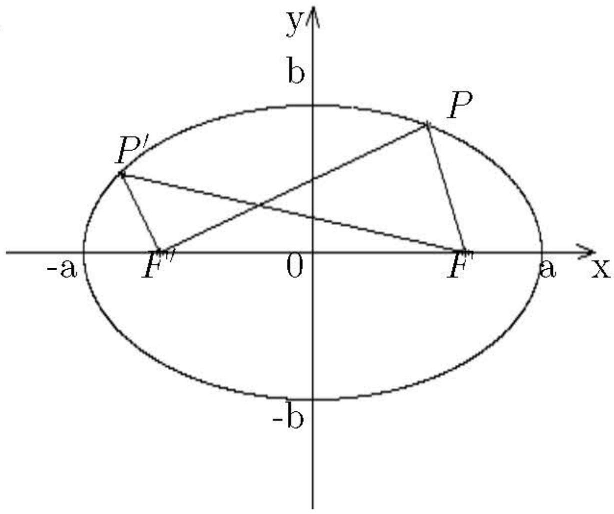
2.5 Ellipses

2.5.1 Definition of Ellipse



An ellipse is the locus of points such that the sum of its distances from two fixed points F and F' is constant.
Points F and F' are each called to be a focus of the ellipse.
The ellipse is symemtric with respect to FF'
The ellipse is symemtric with respect to perpendicular bisector of FF'

2.5.2 Equation of Ellipse



An ellipse is defined as the locus of points such that the sum of its distances from two fixed points F and F' is constant. Points F and F' are each called to be a focus of the ellipse.
The ellipse is symmetric with respect to FF'
The ellipse is symmetric with respect to perpendicular bisector of FF'.

. Choose the origin O to be the midpoint of $F'F$.
. Draw x-axis passing through OF with direction $\overrightarrow{OF}$
. Draw y-axis passing though O and perpendicular to x-axis.
Take $C = OF$ then $F(C, 0)$ and $-C = OF'$ then $F'(-C, 0)$
Let $P(x, y)$ and $FP + F'P = 2a$ (constant, $a > c$).
Then $\sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = 2a$
$\sqrt{(x+c)^2 + y^2} = 2a - \sqrt{(x-c)^2 + y^2}$
$(x+c)^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + (x-c)^2 + y^2 4cx - 4a^2$
$= -4a\sqrt{(x-c)^2 + y^2}$
$(cx - a^2)^2 = (-a\sqrt{(x-c)^2 + y^2})^2$
$c^2x^2 - 2a^2cx + a^4 = a^2\{(x-c)^2 + y^2\}$
$(x^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2)$.
Therefore $\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2 - c^2} = 1.(\star)$
Since $a^2 = b^2 + c^2$, $a > b$, therefore, $c = \sqrt{a^2 - b^2}$.

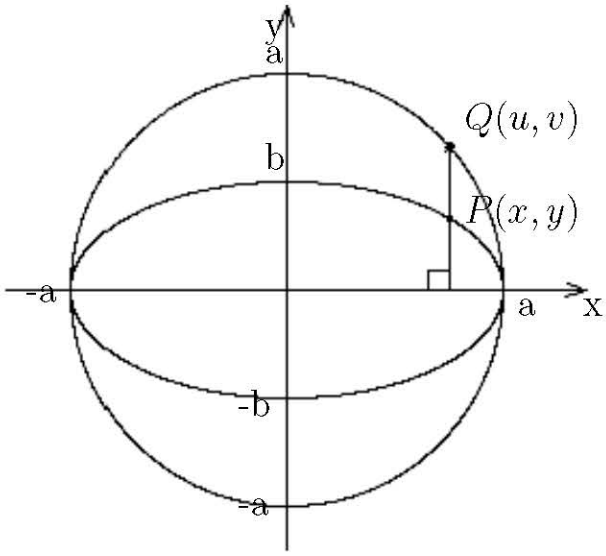
Thus the equation ($\star$) is $\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2} = 1$.
These fixed points $F(\sqrt{a^2-b^2}, 0)$ and $F'(-\sqrt{a^2-b^2}, 0)$ are called foci.
The ellipse which the sum of the distances from foci F and F' is $2a$ is shown as picture above.

2.5.3 Eccentricity

The ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ with the major axis $2a$ is given. The ratio of distance from the center of the ellipse to the focus for a is called eccentricity. Eccentricity is normal denoted e. We have $e = \dfrac{OF}{a} = \dfrac{\sqrt{a^2-b^2}}{a}$, where $0 < e < 1$
If an eccentricity e approaches to 0, then the ellipse becomes smimilar with a circle.
The coordinates of foci are often denoted by $F(ae, 0)$ and $F'(-ae, 0)$.

2.5.4 Circle and Ellipse

If $a > b > 0$, then the circle which has the major axis of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \cdots (1)$ as its diameter is $x^2 + y^2 = a^2 \cdots (2)$ (See the picture).
Take $Q(u, v)$ as a point moving along the circumference of circle, and $P(x, y)$ as a point whose y-coordinate is $\dfrac{b}{a}$ times of the y-coordinate of Q.



Then, $u^2 + v^2 = a^2 \cdots$ (3).
$x = u,\ y = bav$ or $v = \dfrac{a}{b}y$.
Substituting these values into (3), we have
$x^2 + (\dfrac{a}{b}y)^2 = a^2$ or $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.
Thus, point P lies on ellipse (1)
Therefore, ellipse (1) can be considered as defining circle (2) at a ratio of $\dfrac{b}{a}$ along the y-axis.

2.5.5 Polar Coordinate of Ellipse

When two points $Q(a\cos\theta, a\sin\theta)$ on the circle $x^2+y^2=a^2$ and $R(b\cos\theta, b\sin\theta)$ on the circle $x^2+y^2=b^2$ moves to the point $P(x,y)$, the coordinates of P are expressed as: $(x,y)=(a\cos\theta, bsin\theta)$

For an ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, if we put $x=a\cos\theta$, and $y=bsin\theta$, then we have

$$\begin{aligned}\frac{x^2}{a^2}+\frac{y^2}{b^2} &= \frac{a^2\cos^2\theta}{a^2}+\frac{b^2\sin^2\theta}{b^2}\\ &= \cos^2\theta+\sin^2\theta\\ &= 1.\end{aligned}$$

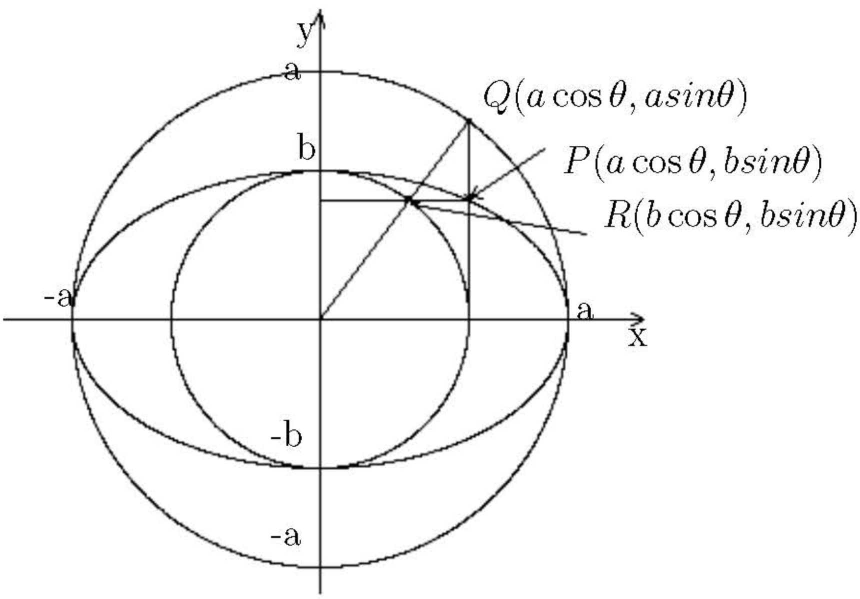


Figure. Coordinate polar of ellipse

2.5.6 Tangent Line of Ellipse

Example. Prove that the equ tion of a tangent line of an lipse is given by the following t froms:

1. The tangents with the slo m are

$$y = mx + \sqrt{a^2m^2 + b^2}$$

2. The tangent at a point $(x_1,$ is:

$$\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1$$

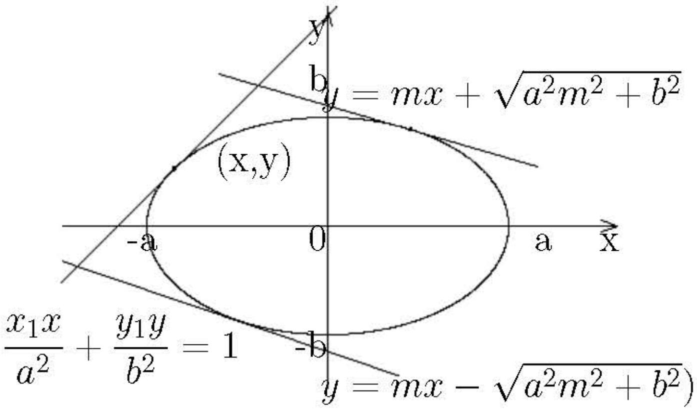


Figure. Tangent line of an ellipse

Answer1. The tangents with the slope m are: $y = mx \pm \sqrt{a^2m^2 + b^2}$

Let $y = mx + c$ as the line which, tangent to an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Substitute $y = mx + c$ into $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ we have $(mx + c)^2 = b^2 - \frac{b^2}{a^2}x^2$

$$(m^2x^2 + 2mxc + c^2)a^2 = a^2b^2 - b^2x^2$$

$$x^2(a^2m^2 + b^2) + 2mxca^2 + c^2a^2 - a^2b^2 = 0$$

$$\begin{aligned}\triangle' &= m^2c^2a^4 - (a^2m^2 + b^2)(c^2a^2 - a^2b^2)\\ &= m^2c^2a^4 - m^2c^2a^4 + m^2a^4b^4 + b^2c^2a^2 + a^2b^4\\ &= m^2a^4b^2 + b^2c^2a^2 + a^2b^4\end{aligned}$$

Since the line tangent to ellipse we have:

$$m^2a^4b^2 - b^2c^2a^2 + a^2b^4 = 0 \Longleftrightarrow c = \pm\sqrt{a^2m^2 + b^2}$$

Therefore the tangent line of ellipse which has slope m is $y = mx \pm \sqrt{a^2m^2 + b^2}$

2. The tangent at a point (x_1, y_1) is $\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1$.

Differentiate both sides of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

we have $2b^2x + 2a^2yy' = 0 \Rightarrow y' = -\frac{b^2x}{a^2y}$.

the slope of line l at a point $P(x_1, y_1)$ is $-\frac{b^2x_1}{a^2y_1}$

When $x_1 \neq 0, y_1 \neq 0$ equation of tangent line is $y - y_1 = -\frac{b^2x_1}{a^2y_1}(x - x_1)$

$$a^2y_1(y-y_1) = -b^2x_1(x-x_1)$$
$$a^2y_1y - a^2y_1^2 = -b^2x_1x + b^2x_1^2$$
$$b^2x_1x + a^2y_1y = a^2b^2 \cdots (\star)$$

(Because $b^2x_1^2 + a^1y_1^2 = a^2b^2$)
Therefore

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1.$$

When $x_1 = 0$, in this case

$$y_1 = \pm b \ (\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1).$$

Therefore the tangent line is: $y = b$ or $y = -b$.
When $y_1 = 0$, in this case

$$x_1 = \pm a \ (\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1).$$

Therefore the tangent line is: $x = a$ or $x = -a$.
Both of two cases above satisfy to equation$\cdots (\star)$
Therefore equation of tangent line at point $P(x_1, y_1)$ of ellipse is:

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1.$$
$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$$

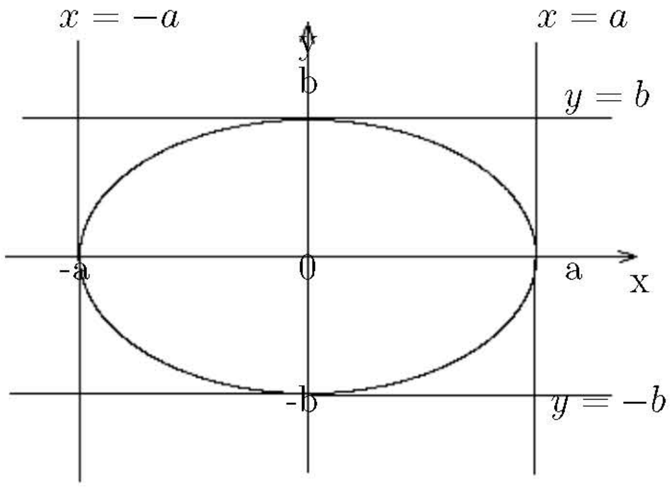


Figure. Tangent lines $y = \pm b$ and $x = \pm a$ of ellipse

2.5.7 Angle of Tangent Line and Ellipse

Example. Given the line $XY$is a tangent of the ellipse at a point P, and F and F' are the foci of the ellipse. Prove that $\angle FPX = \angle F'PY$.

Answer. Put $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a > b > 0)$ as the equation of the ellipse, and $P(x_1, y_1)$.
We have $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$ (i), and the equation of the tangent XY is $\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1$.

Put $x_1 \neq 0$. The intersection point Q of XY and x-axis is $Q(\frac{a^2}{x_1}, 0)$.
If we put $F(c,0), F'(-c,0)$, and $c = \sqrt{a^2-b^2}$, we have
$FQ = |\frac{a^2 - cx_1}{x_1}|$, $F'Q = |\frac{a^2 + cx_1}{x_1}|$
$PF = |\sqrt{(x_1-c)^2 + y_1^2}| = |\frac{cx_1 - a^2}{a}|$
$PF' = |\sqrt{(x_1-c)^2} + y_1^2| = |\frac{cx_1 + a^2}{a}|$
Therefore $\frac{FQ}{F'Q} = \frac{PF}{PF'}$.

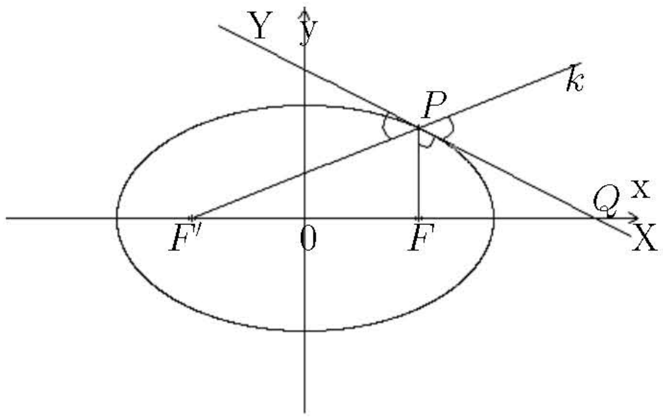


Figure. Angle of tangent line XY and ellipse

By property of an external angle we have:
$\angle FPQ = \angle KPQ$.
since $KPQ = \angle F'PY$.
Thus $\angle FPX = \angle F'PY$.
if $x_1 = 0$, then the point P is on the y-axis,
$PF = PF'$ and $XY//FF'$ (the right figure).
Therefore $\angle FPX = \angle F'PY$.

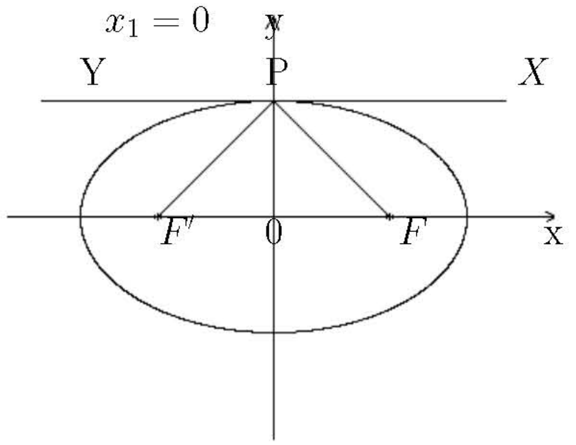


Figure. Angle of tangent line XY and ellipse, when $XY//FF'$

2.6 Hyperbolas

2.6.1 Definition of Hyperbola

A hyperbola is the locus of a point such that the difference of its distances from two fixed points F and F' is a constant, and F and F' are called the **foci**.

A hyperbola is also symentric with respect to the staight line connecting the two foci and the perpendicular bisector of the segment connecting the two foci. Therefore, it is also symmentric with respect to the midpoint O of the line segment connecting the two foci. This point O is referred to as the **center** of the hyperbola.

just as we did for the case of an ellipse, let us define coordinate axes by taking the center O as the origin, the line FF' connecting the two foci as the x-axis, and the perpendicular bisector of line segment FF' as the y-axis.

2.6.2 Equation of Hyperbola

A hyperbola is defined as the locus of points such that the difference of its distances from two fixed points F and F' are called the foci.
Let $F(c, 0)$ and $F'(-c, 0)$ be fixed points and $P(x, y)$ be the point which satisfies $|PF - PF'| = 2a$ (a: is a constant, $c > a > 0$)
Then

$$\begin{aligned}
\sqrt{(x-c)^2+y^2} - \sqrt{(x+c)^2+y^2} &= \pm 2a \\
\sqrt{(x-c)^2+y^2} &= \pm 2a + \sqrt{(x+c)^2+y^2} \\
(x-c)^2+y^2 &= 4a^2 \pm 4a\sqrt{(x+c)^2+y^2} + (x+c)^2+y^2 \\
-4cx - 4a^2 &= \pm 4a\sqrt{(x+c)^2+y^2} \\
(-cx-a^2)^2 &= (\pm a\sqrt{(x+c)^2+y^2})^2 \\
c^2x^2 + 2a^2cx + a^4 &= a^2\{(x+c)^2+y^2\} \\
(c^2-a^2)x^2 - a^2y^2 &= a^2(c^2-a^2).
\end{aligned}$$

As $c^2 - a^2 > 0$ $(c > a)$, $\dfrac{x^2}{a^2} - \dfrac{y^2}{c^2-a^2} = 1.$
If we put $b^2 = c^2 - a^2$, we have $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1 \cdots (\star)$.
The equation $(\star)$ is called the **standard from** of a hyperbola. The points $A(a, 0)$ and $A'(-a, 0)$ are called the **vertices**, and the point $O(0, 0)$ is the **center** of the hyperbola $(\star)$.
Conversely, the locus of points $P(x, y)$ that satisfy the equation $(\star)$ is a hyperbola. That is any point P on the curve $(\star)$ satisfies the identity $|PF - PF'| = 2a$ where $F(\sqrt{a^2+b^2}, 0)$ and $F'(-\sqrt{a^2+b^2}, 0)$.

2.6.3 Asymptotes of Hyperbola

An asymptote is a stright line whose perpendicular distance from a curve decreases to zero as the distance from the origin increases without limit. For example, the hyperbola $xy = 1$ has a vertical and horizontal asymptotes, the hyperbola $x^2 - y^2 = 1$ has 45^o symtotes.

In the hyperola $\frac{(x-x_0)^2}{a^2} - \frac{(y-y_0)^2}{b^2} = 1$, the asymptoes can be found by substituting 0 to the 1 on the right side of the general equation and therefore have slope $\pm\frac{b}{a}$.
That is the equation of the asymptotes is

$$\frac{(x-x_0)^2}{a^2} - \frac{(y-y_0)^2}{b^2} = 0$$

$$\frac{(y-y_0)^2}{b^2} = \frac{(x-x_0)^2}{a^2}$$

$$(y-y_0)^2 = \frac{b^2}{a^2}(x-x_0)^2$$

$$y - y_0 = \pm\frac{b}{a}(x-x_0).$$

Especially, for the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

we have the asymptotes

$$y = \pm\frac{b}{a}x.$$

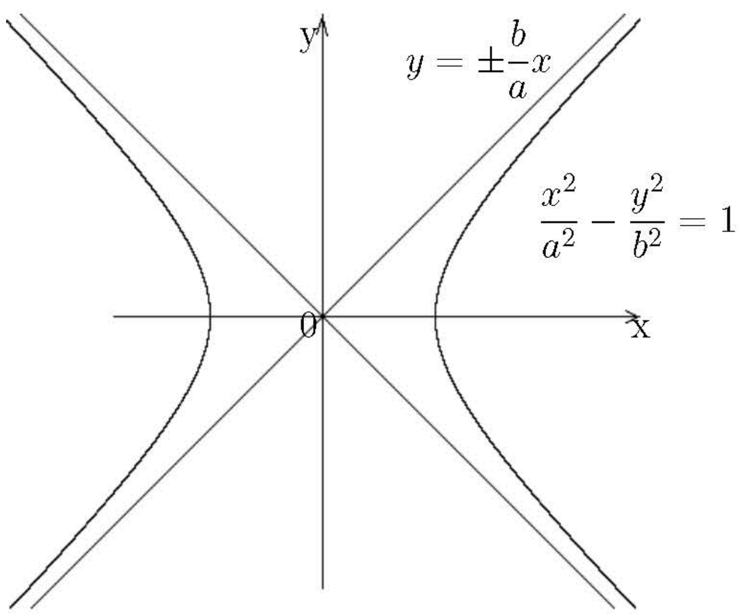
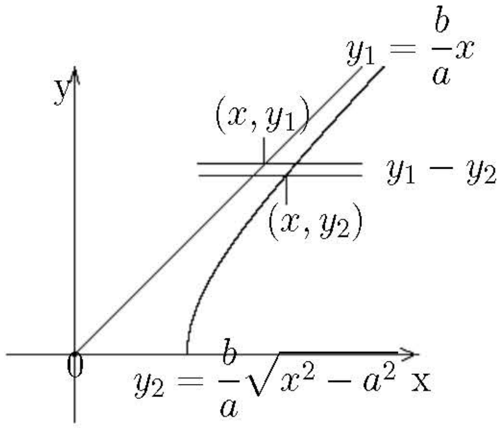


Figure. Hyperbola and its asymptotes

Answer way.

Equation $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ we have
$\pm\dfrac{b}{a}\sqrt{x^2-a^2}$.
By the right figure, we have
$\dfrac{b}{a}\sqrt{x^2-a^2}$, $y_1 = \dfrac{b}{a}x$. When x
∞, we have:



$$\lim_{n\to\infty}\left(\frac{b}{a}x - \frac{b}{a}\sqrt{x^2-a^2}\right) = b\lim_{n\to\infty}($$

$$= b\lim_{n\to\infty}$$

$$= b\lim_{n\to\infty}$$

Similarly for $y_2 = -\dfrac{b}{a}x$.

2.6.4 Tangent Line of Hyperbola

Example.

1. Prove that the tangent
slope is m is given as $y =$
$\sqrt{a^2m^2-b^2}$.

2. Prove that the tange
$P(x_1, y_1)$ on the hyperbola $\dfrac{x^2}{a^2} -$
1 is given as:

$$\frac{x_1x}{a^2} - \frac{y_1y}{b^2} = 1.$$

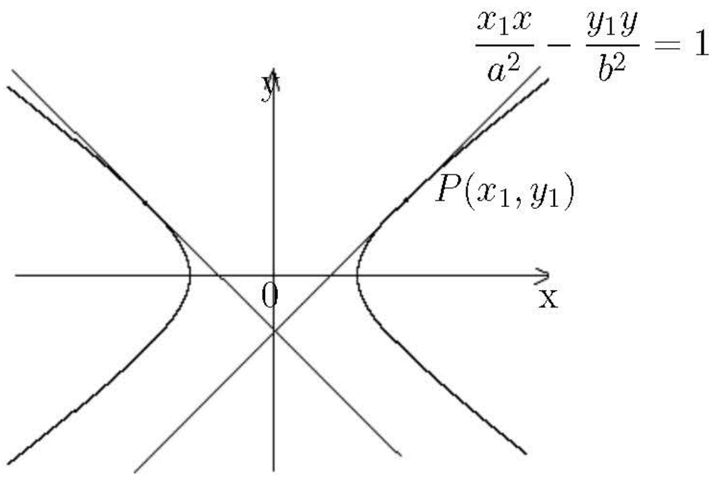


Figure Tangent line of hyperbola

Answer 1. Let $y = mx + c$ be an equation line whose is equal to slope m.

Substitute $y = mx + c$ into $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
we have
$(mx + c)^2 = -b^2 + \frac{b^2}{a^2}x^2$
$(m^2x^2 + 2mxc + c^2)a^2 = -a^2b^2 + b^2x^2$
$x^2(a^2m^2 - b^2) + 2mxca^2 + c^2a^2 + a^2b^2 = 0$
$\triangle' = m^2c^2a^4 - (a^2m^2 - b^2)(c^2a^2 + a^2b^2)$
$= m^2c^2a^4 - m^2c^2a^4 - m^2a^4b^2 + b^2c^2a^2 + a^2b^4$
$= -m^2a^4b^2 + b^2c^2a^2 + a^2b^4$.
Since the line tangent to hyperbola, we have $\triangle' = 0$
$-m^2a^4b^2 + b^2c^2a^2 + a^2b^4 = 0$, therefore $c = \pm\sqrt{a^2m^2 - b^2}$.
Thus the tangent line which has slope m is $y = mx \pm \sqrt{a^2m^2 - b^2}$.

Prove that the tangent at $P(x_1, y_1)$ on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is given as

$$\frac{x_1x}{a^2} - \frac{y_1y}{b^2} = 1.$$

Differentiate both sides of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we have $2b^2x - 2a^2yy' = 0 \Rightarrow y' = \frac{b^2x}{a^2y}$
Slope of line l at point $P(x_1, y_1)$ is $\frac{b^2x_1}{a^2y_1}$.
When $y_1 \neq 0$ equation of tangent line is
$y - y_1 = \frac{b^2x_1}{a^2y_1}(x - x_1)a^2y_1(y - y_1) = b^2x_1(x - x_1)$
$a^2y_1y - a^2y_1^2 = b^2x_1x - b^2x_1^2$
$b^2x_1x - a^2y_1y = b^2x_1^2 - a^2y_1^2$
$b^2x_1x - a^2y_1y = a^2b^2 \cdots (\star)(b^2x_1^2 - a^2y_1^2 = a^2b^2)$.
Therefore $\frac{x_1x}{a^2} - \frac{yy_1}{b^2} = 1$.
When $y_1 = 0$, in this case $x_1 = \pm a(\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1)$.
Therefore the equation of tangent line is: x=a or x=-a.
These cases satisfy equation $(\star)$.
Thus equation of tangent line of hyperbola at point $P(x_1, y_1)$ is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$.

2.6.5 Angle of Tangent Line and Hyperbola

Example. Let F and F' be the foci of a hyperbola and P be a point on the hyperbola.

Prove that the tangent of the hypebola at P bisects $\angle FPF'$.

Answer. Let $P(x_1, y_1)$ be a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ $(a > 0,\ b > 0)$.
Then we have $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \cdots$ (i).
The tangent at P is

$$\frac{x_1 x}{a^2} - \frac{y_1 y}{b^2} = 1 \cdots (ii)$$

Put $y = 0$ in the equation (ii), we have the coordinates of the inters

$$\frac{x_1 x}{a^2} - \frac{y_1 .0}{b^2} = 1$$

$$x = \frac{a^2}{x_1}.$$

Therefore $Q(\frac{a^2}{x_1}, 0)$.

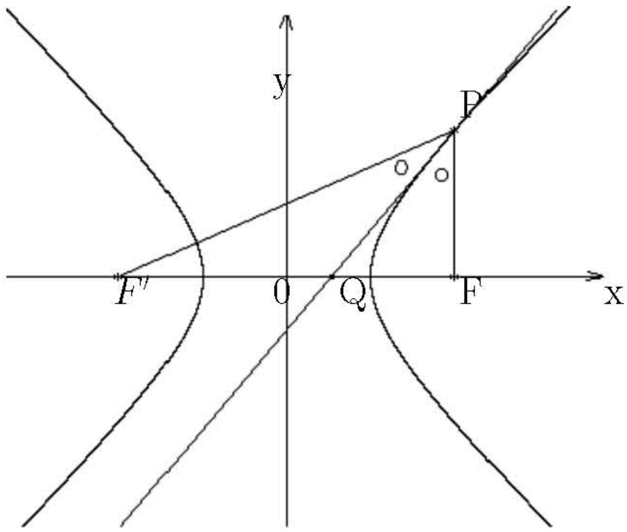


Figure. Angle of tangent line and hyperbola

Put $F(c, 0)$ and $F'(-c, 0)$ where $c = \sqrt{a^2 + b^2}$. Then

$$FQ = |\frac{a^2}{x_1} - c| = |\frac{a^2 - cx_1}{x_1}|$$

$$F'Q = |\frac{a^2}{x_1} + c| = |\frac{a^2 + cx_1}{x_1}|$$

$$F = \sqrt{(c - x_1)^2 + y_1^2} = |\sqrt{c^2 - 2cx_1 + x_1^2 + y_1^2}|$$

$$= \sqrt{a^2 + b^2 - 2cx_1 + x_1^2 + \frac{b^2}{a^2}x_1^2 - b^2} = \sqrt{a^2 - 2cx_1 + \frac{c^2}{a^2}x_1^2}$$

$$= \sqrt{(a - \frac{cx_1}{a})^2} = |a - \frac{cx_1}{a}|.$$

Similarly $PF' = |\frac{a^2 + cx_1}{a}|$

Therefore in $\triangle PFF'$, we have $\frac{FQ}{F'Q} = \frac{PF}{PF'} = |\frac{a^2 - cx_1}{a^2 + cx_1}|$.

Thus we have $\angle FPQ = \angle F'PQ$.
Therefore the tangent at P bisects $\angle FPF'$.

2.6.6 Generalization

Conics may be defined over other fields, and may also be classified in the projective plane rather than in the affine plane.
Over the complex numbers ellipses and hyperbolas are not distinct, since there is no meaningful difference between 1 and -1; precisely, the ellipse $x^2 + y^2 = 1$ becomes a hyperbola under the substitution $y = iw$, geometrically a complex rotation, yielding $x^2 - w^2 = 1a$ hyperbola is simply an ellipse with an imaginary axis length. Thus there are two ways of classifications ellipse, hyperbola and parabola. Geometrically, this corresponds to intersecting the line at infinity in either two distinct points (corresponding to two asymptotes) or in one double point (corresponding to the axis of a parabola), and thus the real hyperbola is a more suggestive image for the complex ellipse, hyperbola, as it also has two (real) intersections with the line at infinity.
In projective space, over either the reals or complex numbers, all non-degenerate conics are equivalent, and thus in projective geometry one simply speaks of " a conic " without specifying a type, as type is not meaningful. Geometrically, the line at infinity is no longer special (distinguished), so while some conics intersect the line at infinity differently, this can be changed by a projective transform pulling an ellipse out to infinity or pushing a parabola off infinity to an ellipse or a hyperbola.

Chapter 3

Transformation

3.1 Introduction

In mathematics, a transformation could be any function mapping a set X on to another set or on to itself. However, often the set X has some additional algebraic or geometric structure and the term "transformation" refers to a function from X to itself which preserves this structure.

Transformation is included ***linear transformations and affine transformations, rotations, reflections and translations***. These can be carried out in Euclidean space, particularly in dimensions 2 and 3. They are also operations that can be performed using linear algebra, and described explicitly using matrices.

3.2 Isometries:

Things which coincide with one another are equal to one another. ——Euclid: "Elements" book I, Common Notion 4

"A figure coincides with another" means that the former coincides with the latter by a "motion", which is a transformation which does not change size or shape. Sometimes, shape or size may be changed, but some properties remain unchanged.

Let us formulate the fact mathematically.
In this sequel, we consider transformations of points on a plane.

Definition: Transformation which does not change the distance between any two points is called a **motion**, an **isometry** or a **congruent transformation.**

Example: (parallel) **transformation, rotation and reflection** are isometries.

Theorem: Isometries preserve each of the followings:
(1) length of segment, length of curve (2) magnitude of angle (3) area of figure etc.

Definition: If a figure X coincides with a figure Y by some isometry, we say that X is **congruent** to Y, and denote the fact as $X \equiv Y$.

Corollary: If two figures are congruent to each other, each of the following is common to them as far as it has meaning:
(1) length (2) magnitude of angle (3) area etc.

3.3 Similarity Transformations

Definition: A similarity transformation is a transformation which takes each segment into a segment whose length is a constant multiple of the length of the original segment. The constant number is called the ratio of magnification.

Example: dilatation is a similarity transformation.
A similarity transformation with ratio of magnification 1 is nothing but an isometry. Thus, "similarity" is more general then "isometry".

Theorem: Similarity transformations preserve each of the followings:
(1) ratio of lengths of two segments (2) magnitude of angle (3) ratio of areas of two figures etc.

Definition: If a figure X coincides with a figure Y by some similarity transformation, we say that X is **similar** to Y, and denote the fact as $X \sim Y$.

Corollary. If two figures are similar to each other, each of the followings is common to them as far as it has meaning:

(1) ratio of lengths of two segments
(2) magnitude of angle
(3) ratio of areas of two parts ect.

3.4 Linear Transformations

A linear map, linear transformation, or linear operator (in some contexts also called linear function) is a function between two vector spaces that preserves the operations of vector addition and scalar multiplication. The expression "linear operator" is commonly used for linear maps from a vector space to itself (endomorphisms). In advanced mathematics, the definition of linear function coincides with that of linear map, while in analytic geometry it is less strict.

3.4.1 Definition:

Let V and W be vector spaces over the same field K. A function $f : V \longrightarrow W$ is said to be a *linear map*. If for any two vectors x and y in V and any scalar α in K, the following two conditions are satisfied:

$f(\overrightarrow{x} + \overrightarrow{y}) = f(\overrightarrow{x}) + f(\overrightarrow{y}))$ additivity

$f(\alpha\overrightarrow{x}) = \alpha f(\overrightarrow{x})$ homogeneity of degree 1.

This is equivalent to requiring that for any vectors $x_1, \cdots, x_m \in V$ and scalars $a_1, \cdots a_m \in K$, the following equality holds: $f(a_1\overrightarrow{x_1} + \cdots + a_m\overrightarrow{x_m}) = a_1 f(\overrightarrow{x}_1) + \cdots + a_m f(\overrightarrow{x}_m)$.

It immediately follows from the definition that $f(0) = 0$.

Occasionally, V and W can be considered to be vector spaces over different fields. It is then necessary to specify which of these ground fields is being used in the definition of "linear". If V and W are considered as spaces over the field K as above, we talk about K-linear maps. For example, the conjugation of complex number is an $\mathbf{R}$-linear map $\mathbf{C} \longrightarrow C$, but it is not $\mathbf{C}$-linear.

A linear map from V to K (with K viewed as a vector space over itself) is called a linear functional.

Examples:

. The identity map and zero map are linear.

. The map $x \longmapsto cx$, where c is a constant, is linear.

. For real numbers, the map $x \longmapsto x^2$ is not linear.

. For real numbers, the map $x \longmapsto x + 1$ is not linear (but is an affine transformation, and also a linear function, as defined in analytic geometry.)

. If A is a real $m \times n$ matrix, then A defines a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$

by sending the column vector $x \in \mathbb{R}^n$ to the column vector $Ax \in \mathbb{R}^m$. Conversely, any linear map between finite-dimensional vector spaces can be represented in this manner, see the following section.

. The integral is a linear map from the space of all real-velued integrable functions on some interval to $\mathbb{R}$.

. Differentiation is a linear map from the space of all differentiable functions to the space of all functions.

. If V and W are finite-dimensional vector spaces over a field F, then functions that send linear maps $f: V \longrightarrow W$ to $\dim_F(W)$-by-$\dim_F(V)$ matrices in the way described in the sequel are themselves linear maps.

. The expected value of a random variable X is linear, as $E[cX + a] = cE[X] + a$, but the variance of a random variable is not linear, as it violates the second condition, homogeneity of degree 1: $v[cX + a] = c^2V[X]$.

3.4.2 Transformation Matrix

If V and W are finite-dimensional, and one has chosen bases in those spaces, then every linear map from V and W can be represented as a matrix; this is useful because it allows concrete calculations.
Conversely, matrices yield examples of linear maps: if A is a real m-by-n matrix, then the rule $f(x) = Ax$ describes a linear map $\mathbb{R}^n \longmapsto \mathbb{R}^m$.

Let $\{\overrightarrow{v}_1, \cdots, \overrightarrow{v}_n\}$ be a basis for V. Then every vector v in V is uniquely determined by the coefficients $c_1, \cdots, c_n$ in $c_1\overrightarrow{v}_1 + \cdots, +c_n\overrightarrow{v}_n$.
If $f: V \longmapsto W$ is a linear map,

$$f(c_1\overrightarrow{v}_1 + \cdots + c_n\overrightarrow{v}_n) = c_1 f(\overrightarrow{v}_1) + \cdots + c_n f(\overrightarrow{v}_n),$$

which implies that the function f is entirely determined by the values of $f(\overrightarrow{v}_1), \cdots, f(\overrightarrow{v}_n)$.
Now let $\{\overrightarrow{w}_1, \cdots, \overrightarrow{w}_m\}$ be a basis for W. Then we can represent the values of each $f(\overrightarrow{v}_j)$ as

$$f(\overrightarrow{v}_j) = a_{1j}\overrightarrow{w}_1 + \cdots + a_{mj}\overrightarrow{w}_m.$$

Thus, the function f is entirely determined by the values of a_{ij}.
If we put these values into an m-by-n matrix M, then we can conveniently use it to compute the value of f for any vector in V. For if we place the value of $c_1, \cdots, c_n$ in an n-by-1 matrix C, we have $MC = the\ m - by - 1\ matrix$ whose i.th element is the coordinate of $f(v)$ which belongs to the base $\overrightarrow{W}_i$.

A single linear map may be represented by many matrices. This is because the values of the elements of the matrix depend on the bases that are chosen.

Example:
Some special cases of linear transformations of two-dimensional space $\mathbb{R}^2$ are illuminating:

. rotation by 90 degrees counterclockwise:

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

. rotation by θ degrees counterclockwise:

$$A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

. reflection against the x-axis:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

. reflection against the y-axis:

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

. scaling by 2 in all directions:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

. horizontal shear mapping:

$$A = \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}$$

. squeezing:

$$A = \begin{bmatrix} k & 0 \\ 0 & \frac{1}{k} \end{bmatrix}$$

. projection onto the y-axis:

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

3.4.3 Forming New Linear Maps From Given One

The composition of linear maps is linear: if $f : V \longmapsto W$ and $g : W \longmapsto Z$ are linear, then so is their composition $gof : V \longmapsto Z$. It follows from this that the class of all vector spaces over a given field K, together with K-linear maps as morphisms, forms a category.

The inverse of a linear map, when defined, is again a linear map.
If $f_1 : V \longmapsto W$ and $f_2 : V \longmapsto W$ are linear, then so is their sum $f_1 + f_2$ (which is defined by $(f_1 + f_2)(x) = f_1(x) + f_2(x)$).

If $f : V \longmapsto W$ is linear and a is an element of the ground field K, then the map af, defined by $(af)(x) = a(f(x))$, is also linear.

Thus the set $L(V, W)$ of linear maps from V to W itself forms a vector space over K, sometimes denoted $Hom(V, W)$. Furthermore, in the case that$V = W$, this vector space (denoted $End(V)$) is an associative algebra under composition of maps, since the composition of two linear maps is again a linear map, and the composition of maps is always associative. This case is discussed in more detail below.

Given again the finite-dimensional case, if bases have been chosen, then the composition of linear maps corresponds to the matrix multiplication, the addition of linear maps corresponds to the matrix addition, and the multiplication of linear maps with scalars corresponds to the mutiplication of matrices with scalars.

3.5 Affine Transformation

An affine transformation or affine map or an affinity (from the Latin, affinis, "connected with") between two vector spaces (strictly speaking, two affine spaces) consists of a linear transformation followed by a translation:

$$x \mapsto Ax + b.$$

In the finite-dimensional case each affine transformation is given by a matrix A and a vector b, satisfying certain properties described below.

Geometrically, an affine transformation in Euclidean space is one that preserves

1. The collinearity relation between points; i.e., the points which lie on a line continue to be collinear after the transformation
2. Ratios of distances along a line; i.e., for distinct collinear points p_1, p_2, p_3, the ratio $|p_2 - p_1|/|p_3 - p_2|$ is preserved.

In general, an affine transformation is composed of linear transformations (rotation, scaling or shear) and a translation (or "shift"). Several linear transformations can be combined into a single one, so that the general formula given above is still applicable.

In the one-dimensional case, A and b are called, respectively, slope and intercept.

Definition: An **affine transformation** is a transformation which takes threee collinear points into three collinear points without changing the ratio of distances between them.
A similar transformation is an affine transformation. But the coverse does not hold.

Example: **Parallel projection** (projection by parallel beams) is an affine transformation.

Theorem: Affine transformations preserve each of the followings:

(1) collinearity (for three points, being on one line)
(2) parallelism (for two lines, being parallel)
(3) ratio of division of a segment
(4) ratio of areas of two figures etc.

Definition: If a figure X coincides with a figure Y by some affine transformation, we say that X is **affinely isomorphic** to Y, and denote the fact as $X \overset{\text{A}}{\cong} Y$, temporarity.

Example:
(1) Ractangle is affinely isomorphic to a parrallelogram, and vice versa.
(2) Circle is affinely isomorphic to an an ellipse, and vice versa.
(3) Triangles are affinely isomorphic to one another.

Corollary: If two figures are affinely isomophic to each other, each of the following is common to them as far as it has meaning:

(1) collinearity
(2) parallelism
(3) ratio of division of a segment
(4) ratio of areas of two parts ect.

Example:
(1) A quadrilateral which is not a parallelogram is not affinely isomorphic to a parallelogram, because of parallelism.
(2) A circle is not affinely isomophic to a polygon, because of collinearity.
(3) A parabola is not affinely isomorphic to a circle, because of property.

3.6 Properties of Affine Transformation:

Property1: The image of a line under affine transformation is a line.
i.e,. f : affine tranformation, ℓ : line, then $f(\ell)$: line.
Property2: Affine transformation preserves the ratio of division.
i.e,. f : affine transformation, A, B, C : are colinear with three points, then $\overline{f(A)f(B)} : \overline{f(B)f(C)} : \overline{AB} : \overline{BC}$.
Property3: Affine transformation preserve parallelism.
i.e,. f : is affine transformation, ℓ_1, ℓ_2 : are lines $\ell_1 \parallel \ell_2$, then $f(\ell_1) \parallel f(\ell_2)$.

Proof1: Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be affine transformation, then using some invertible matrix A and some vector P, we have
$f(X) = A(X) + P$ for $\forall X \subset \mathbb{R}^2$. $(\star)$
Let ℓ be a line $\subset \mathbb{R}^2$
then, using a point $X_0 \in \mathbb{R}^2$, vector $V \in \mathbb{R}^2$, we have
$\ell = \{X \mid X = X_0 + tV \qquad (\exists t \in \mathbb{R})\}$
then $f(\ell) = \{f(X) \mid X \in \ell\} = \{f(X_0 + tV \mid t \in \mathbb{R})\}$
By $(\star)$, $f(X) = A(X_0 + tV) + P = AX_0 + A(tV) + P = AX_0 + t(AV) + P = X'_0 + tV'$.
where $X'_0 = AX_0 + P; \quad V' = AV$.
Therefore $f(\ell) = \{X' \mid X' = X'_0 + tV' \quad ()\exists t \in \mathbb{R}\}$.
This set is the line passing through X'_0 with direction vector V'.

Proof2: Let f as above $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be affine transformation.
Let vector $a = \overrightarrow{OA}$, vector $b = \overrightarrow{OB}$, vector $c = \overrightarrow{OC}$
then $C = a + \overrightarrow{AC} = a + t(b - a)$

then

$$\begin{aligned} f(C) &= f(a + t(b-a)) \\ &= A(a + t(b-a)) + P \\ &= Aa + tA(b-a) + P \\ &= Aa + t(Ab + Aa) + P \\ f(A) &= f(a) = Aa + P \qquad (A : \quad Matrix) \end{aligned}$$

Therefore $\overrightarrow{f(A)f(C)} = f(C) - f(a) = t(Ab - Aa)$
Likewise

$$\begin{aligned} \overrightarrow{f(A)f(B)} &= f(b) - f(a) \qquad\qquad = Ab + P - (Aa + P) \\ &= Ab - Aa \end{aligned}$$

Hence, $\overrightarrow{f(A)f(C)} = t\overrightarrow{f(A)f(B)}$.
Thus, $\overline{f(A)f(B)} : \overline{f(B)f(C)} = 1 : (t-1)$
Similarly, from $\overrightarrow{AC} = t\overrightarrow{AB}$, we have $AB : BC = 1 : (t-1)$
Therefore $\overline{f(A)f(B)} : \overline{f(B)f(C)} = \overline{AB} : \overline{BC}$.

Proof3: Let f as above $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be affine transformation.
Let ℓ_1, ℓ_2 be two parallel lines
then, we can express
$\ell_1 = \{X \mid X = X_1 + tV_1 \quad (\exists t \in \mathbb{R})\}$
$\ell_2 = \{X \mid X = X_2 + tV_2 \quad (\exists t \in \mathbb{R})\}$
Let $V' = AV$ (as in the proof1), then we have

$$\begin{aligned} f(\ell_1) &= \{X' \mid X' = X_1' + tV' \qquad (\exists t \in \mathbb{R})\} \\ f(\ell_2) &= \{X' \mid X' = X_2' + tV' \qquad (\exists t \in \mathbb{R})\} \end{aligned}$$

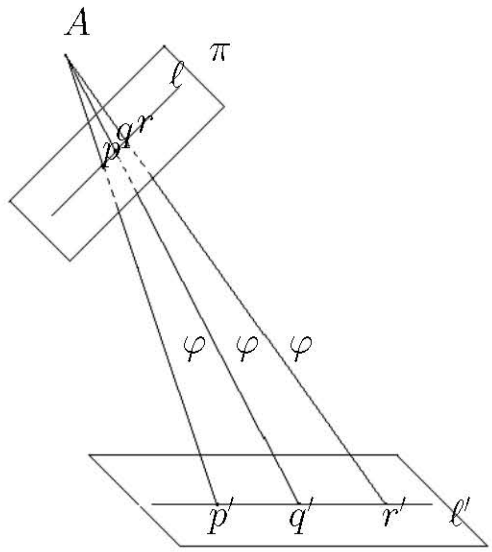
Therefore $f(\ell_1) \parallel f(\ell_2)$.

Definition
Property or quantity on geometric figure is said to be affinely invariant if it is preserved by any affine transformation. (p or q is called an affine invariant if it is preserved by affine transformation).
Example:
1)- Parallelism (= being parallel) is an affine invariant property.
2)- Ratio of length of parallel seqment is an affine invariant quantity.
3)- Perpendicularity is not affinely invariant.
4)- Ratio of length of two non-parallel seqments are not affinely invariant.

3.7 Projective Transformation:

Definition:

Given two plans π, π' and a point A, then we can define a mapping.
$\varphi = \pi \longrightarrow \pi'$
by $\pi \longmapsto P' = AP \cap \Pi'$, where AP is a whole line (NOT a seqment).
Therefore φ is called the perspective mapping from π to π' with center A.



Definition: A projective tranformation of $\mathbb{R}^2$ is a mapping f defined as

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \dfrac{a_1x + a_2y + a_3}{c_1x + c_2y + c_3} \\ \dfrac{b_1x + b_2y + b_3}{c_1x + c_2y + c_3} \end{pmatrix}, \text{ where } \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \neq 0$$

Example:

Let $\begin{pmatrix} x' \\ y' \end{pmatrix} = f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \dfrac{a_1x + a_2y + a_3}{c_1x + c_2y + c_3} \\ \dfrac{b_1x + b_2y + b_3}{c_1x + c_2y + c_3} \end{pmatrix}.$

Let $\begin{pmatrix} x'' \\ y'' \end{pmatrix} = g\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \dfrac{p_1x' + p_2y' + p_3}{r_1x' + r_2y' + r_3} \\ \dfrac{q_1x' + q_2y' + q_3}{r_1x' + r_2y' + r_3} \end{pmatrix}.$

Express x'', y'' in terms of x, y

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = g\begin{pmatrix} x' \\ y' \end{pmatrix} = g\left(f\begin{pmatrix} x \\ y \end{pmatrix}\right) = g\begin{pmatrix} \dfrac{a_1x + a_2y + a_3}{c_1x + c_2y + c_3} \\ \dfrac{b_1x + b_2y + b_3}{c_1x + c_2y + c_3} \end{pmatrix}, \text{ then}$$

$$x'' = \frac{\alpha_1x + \alpha_2y + \alpha_3}{\beta_1x + \beta_2y + \beta_3}\frac{\beta_1x + \beta_2y + \beta_3}{\gamma_1x + \gamma_2y + \gamma_3}$$

where

$\alpha_1 = p_1a_1 + p_2b_1 + p_3c_1$

$\alpha_2 = p_1a_2 + p_2b_2 + p_3c_2$
$\alpha_3 = p_1a_3 + p_2b_3 + p_3c_3$
$\gamma_1 = r_1a_1 + r_2b_1 + r_3c_1$
$\gamma_2 = r_1a_2 + r_2b_2 + r_3c_2$
$\gamma_3 = r_1a_3 + r_2b_3 + r_3c_3$
$\beta_1 = q_1a_1 + q_2b_1 + q_3c_1$
$\beta_2 = q_1a_2 + q_2b_2 + q_3c_2$
$\beta_3 = q_1a_3 + q_2b_3 + q_3c_3$

$$\begin{bmatrix}\alpha_1 & \alpha_2 & \alpha_3\\ \beta_1 & \beta_2 & \beta_3\\ \gamma_1 & \gamma_2 & \gamma_3\end{bmatrix} = \begin{bmatrix}p_1 & p_2 & p_3\\ q_1 & q_2 & q_3\\ r_1 & r_2 & r_3\end{bmatrix}\begin{bmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{bmatrix}$$

It mean that, coefficient matrix of composition of mapping $g \circ f$ =(coefficient matrix of g)$\cdot$ (coefficient matrix of f).

Lemma:

1). Composition of two projective transformations is a projective transformation.

2). Identify tranformation I is a projective transformation.

3). Inverse of a projective tranformation is a projective tranformation.

Proof: **(1)** Suppose f is a projective transformation with coeffication matrix A.

Suppose g is a projective tranformation with coeffication matrix B.

then as already shown composition $g \circ f$ is projective transformation with coefficient matrix BA.

(2) $I\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix}$.

$$I\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\dfrac{1x+0y+0}{0x+0y+1}\\ \dfrac{0x+1y+0}{0x+0y+1}\end{pmatrix} \text{ then } \begin{vmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{vmatrix} = 1 \neq 0$$

So I is a projective transformation with coefficient matrix $\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}$.

(3) Let f be a projective tranformation with coefficient of matrix A, then $detA \neq 0$, when A^{-1} exists.

Let g be a projective transformation with coefficient matrix A^{-1} the $g \circ f$ is a projective transformation with coefficient matrix $A^{-1}A = E$.

Therefore $g \circ f = I$.

Similarly $f \circ g = I$.

Thus, $f^{-1} = g$.
Therefore f^{-1} is a projective transformation.

Definition: A projective transformation is a transformation which preserves collinearity.
N.B As far as projective transformations are concerned, we take the points at infinity in consideration.

An affine tranformation is a projective transformation, but the converse does not hold.

Example: A projection by meams from a **single point** is a projective transformation.

Theorem: Projective transformations preserve each of the followings:

(1) colinearity
(2) concurrency (for three lines, passing through a common point)
(3) ratio $\cdots$ etc.

Definition: If a figure X coincides with figure Y by some projective tranformation, we we say that X is **projective isomophic** to Y, and denote the fact as $X \overset{\mathrm{P}}{\cong} Y$, temporarity.

Example:
(1) A square is projectively isomophic to a trapezoid, and vice versa. (In fact, quadrilaterals are projectively isomorphic to one another.)
(2) A circle is projectively isomophic to a parabola, and vice versa.

Corollary: If two figures are projectively isomorphic to each other, each of the followings is common to them as far as it has meaning:
(1) collinearity (2) concurrency (3) ratio of $\cdots$ (?) etc.

Example:
(1) A traingle is not projectively isomorphic to a quadrilateral, because of collinearity.
(2) Is the pantagon projectively isomorphic to a regular pentagon?
(3) Is the graph of the function $y = x^2$ projectively isomorphic to that of the function $y = x^3$?

3.8 Various Geometries

Given a group of transformations of a space, study the properties on figures in the space which are not changed by the transformations in the group. —
—-Filix Klein(1872)

Various properties and metrics have been defined on figures.

The study of properties and matrics which remain unchanged by isometrics is called **Euclidean geometry**.

The study of properties and metries which remain unchanged by projective transformations is called **projective geometry**.

	Euclidean	"similar"	affine	projective
location	×	×	×	×
distance	○	×	×	×
length				
ratio of distance between 3 points	○	○	×	×
ratio of devision of seqment				
magnitude of angle				
ratio of area of two figures	○	○	○	×
being symmetric with respect to a point				
being symmetric with respect to a line				
collinearity				
concurrency				
parallelism				
perpendicularity				
smoothness (differentiability)				
connectedness				

3.9 Discovering Theorem:

Identifying two different figures sometimes leads us to discovery.

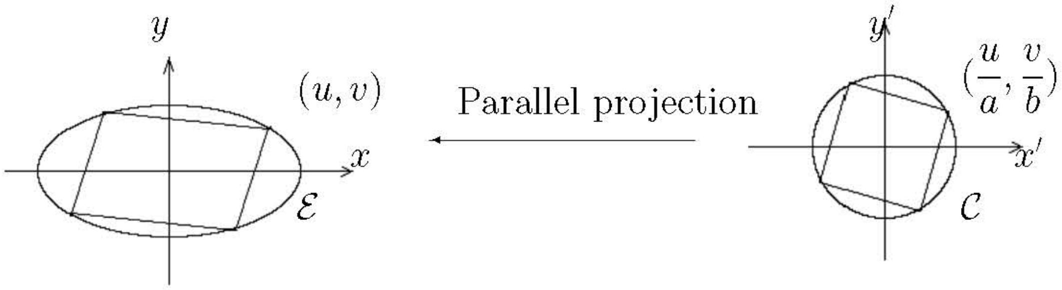
3.9.1 By Affine Transformation:

Example: Amoung quadrilateral inscribed in the ellipse $\mathcal{E}$: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, which has maximum area?

Rhombus with vertices on coordinate axes? Rectangle with sides parallel to coordinate axes?
Let us specialize the problem. In the case $a = b$, the solution will be easily found.

Theorem: Amoung quadrilaterals inscribe in a circle, squares have maximum area.

By an affine transformation the above theorem can be reduced to this theorem. The general situation in the problem can be reguarded as shadows of a circle and inscribed quardrilaterals drawn on the glass of a window.



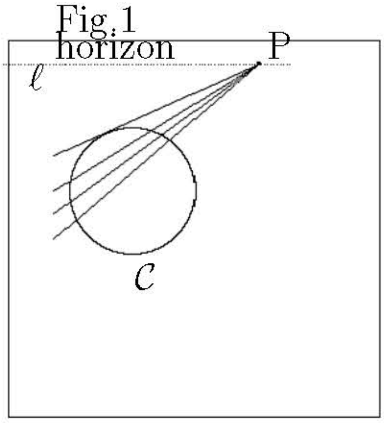
Then, the theorem above and the " invariance of the ratio of areas" in the theorem of affine transformation imply that the solutions are those which are obtained as a shadow of squares inscribed in the circle on the class of the window. Therefore, there are many solutions, not only a rhombus and a rectangle. The solutions are parallelograms, since the shadow of square by sun beam is a parallelogram. But not all parallelograms inscribed in a ellipse have the same area.

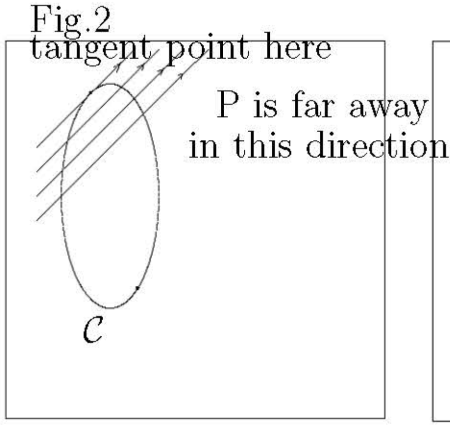
3.9.2 By Projective Projections—How to draw tangent lines

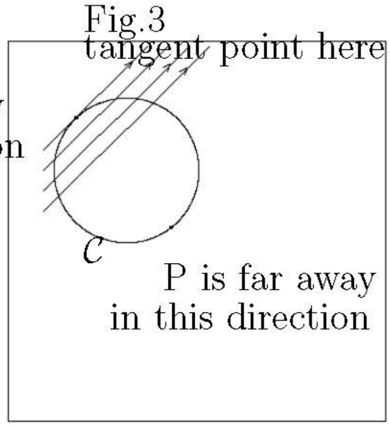
Pay attention to geometric properties common to object and picture. Then we may be able to find some unexpected facts.

Example: Given a circle $\mathcal{C}$ and a point P outside it. How can we draw tangent line frome P to $\mathcal{C}$ using only the ruler.

Suppose the tangent line have been drawn (Fig.1). Consider that ***the figure is a sketch of a figure drawn on the ground, which ℓ being the horizon***. Then what figure must be drawn on the ground. Imagine we look it down from the sky (Fig.2).







On the ground $\mathcal{C}$ must be an ellipse. Transform the ellipse into a circle by affine transformation (Fig.3)

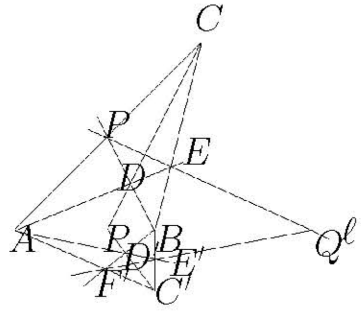
Example: How can we find the tangent points in Fig.3? (They can be found using only a ruler, by means of drawing parallel lines.)

The method will be applicable in Fig.1, since “ being parallel in Fig.3 ” mean “ meeting on the horizon ℓ ”.

Generalization: Remember a parabola that circle can be obtained as a sketch of a parabola drawn on the ground.

3.9.3 By Projective Transformation —-Making a scale:

Theorem: Given a line ℓ and three points A, B, P on it. Choose a point C not on ℓ, a point D on the line PC. Denote the intersection of AC and BC denote by E, the intersection of BD and AC by F, and the intersection FE and ℓ by Q. Then,



$$(\star) \qquad \frac{AP}{PB} = -\frac{AQ}{BQ}$$

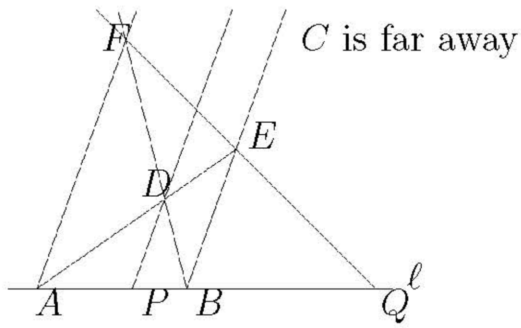
hold. Therefore, Q does not depend on the choice of C and D.

NB. Here the fractions should be signed. For example, $\frac{AP}{PB}$ is positive or negative according as $\overrightarrow{AP}$ and $\overrightarrow{PB}$ are in the same direction or in the opposite direction. Therefore, the above equation ($\star$) implies P divides AB internally [or externally] in the same ratio as Q divides AB externally [or internally].

Such a sequence A, B, P, Q of points is called a **harmonic range of points**, and Q [or P] is called the **harmonic conjugate** of P [or Q] with respect to AB.

Proof of Theorem:

Regard the figure as a sketch, where C is on the horizon and ℓ is drawn to be parallel to the horizon. Looking the figure down from the sky, we will see a parttern as in the figure on the right.
Since ℓ is down parallel to the horizon, the ratio $\frac{AP}{PB}$ on the sketch is equal to the ratio on the ground.
Using the properties of parallel lines and simila traingles, it is not hard to show that ($\star$) holds in the figure on the right $Q.E.D.$



3.10 Rotation

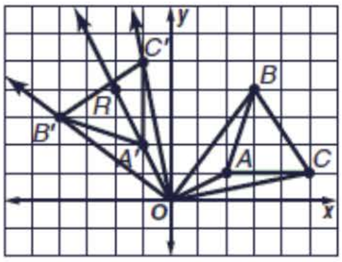
A rotation is a transformation in a plane or in space that describes the motion of a rigid body around a fixed point. A rotation is different from a translation, which has no fixed points, and from a reflection, which "flips" the bodies it is transforming. A rotation and the above-mentioned transformations are isometries; they leave the distance between any two points unchanged after the transformation.
It is important to know the frame of reference when considering rotations, as all rotations are described relative to a particular frame of reference. In general for any orthogonal transformation on a body in a coordinate system there is an inverse transformation which if applied to the frame of reference results in the body being at the same coordinates. For example in two dimensions rotating

a body clockwise about a point keeping the axes fixed is equivalent to rotating the axes counterclockwise about the same point while the body is kept fixed.

3.10.1 Draw Rotation:

A transformation called a rotation turns a figure through a specified angle about a fixed point called the center of rotation. To find the image of a rotation, one way is to use a protractor. Another way is to reflect a fingure twice, in two intersecting lines.

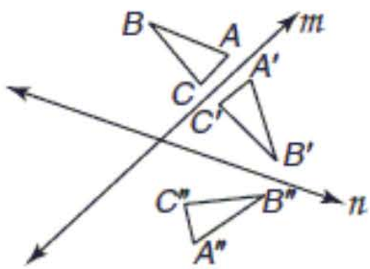
Example 1:
ΔABC has vertices $A(2,1), B(3,4)$, and $C(5,1)$. Draw the image of ΔABC under a rotation of 90^o counterclockwise about the origin.



. First draw ΔABC. Then draw a segment from O, the origin, to point A.
. Use a protractor to measure 90^o counterclockwise with $\overline{OA}$ as one side.
. Draw $\overrightarrow{OR}$.
. Use a compass to coppy $\overline{OA}$ onto $\overrightarrow{OR}$. Name the segment $\overline{OA'}$.
. Repeat with segments from the origin to points B and C.

Example 2:
Find the image of ΔABC under reflection in line m and n.

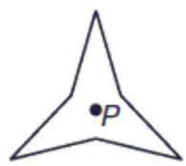
First reflect $\Delta A'B'C'$ in line n. Label the image $\Delta A'B'C'$.
Reflect $\Delta A'B'C'$ in line n. Label the image $\Delta A''B''C''$.
$\Delta A''B''C''$ is a rotation of ΔABC. The center of rotation is the intersection of line m and n. The angle of rotation is twice the measure of the acute angle formed by m and n.



3.10.2 Rotational Symmetry:

When the fingure at the right is rotated about point P by 120^o, the image looks like the preimage. The figure has rotational symmetry, which means it can be rotated less than 360^o about a point and the preimage and image appear to be the same.

The figure has rotational symmetry of order 3 becuase there are 3 rotations less than 360^o $(0^o, 120^o, 240^o)$ that produce an image that is the same as the original. The magnitude of the rotational symmetry for a figure is 360 degrees divided by the order. For the figure above, the rotational symmetry has magnitude 120 degrees.



Example

Identify the order and magnitude of the rotational symmetry of the design at the right.

The design has rotational symmetry about the center point for rotations of 0^o, 45^o, 90^o, 135^o, 180^o, 225^o, 270^o, and 135^o. There are eight rotations less than 360 degrees, so the order of its rotational symmetry is 8. The quotient $360 \div 8$ is 45, so the magnitude of its rotational symmetry is 45 degrees.

3.11 Reflection

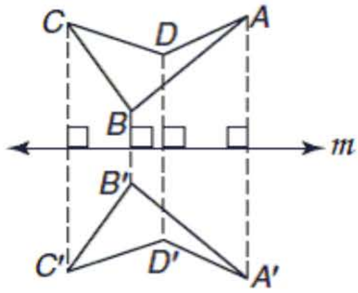
A reflection is a map that transforms an object into its mirror image. For example, a reflection of the small English letter p in respect to a vertical line would look like q. In order to reflect a planar figure one needs the "mirror" to be a line (" axis of reflection"), while for reflections in the three-dimensional space one would use a plane for a mirror. Reflection sometimes is considered as a special case of inversion with infinite radius of the reference circle. Or in easier terms a translation is on coordinate grid you slide the figure over on to another coordinate plane.

3.11.1 Draw Reflections

The transformation called a reflection is a flip of a figure in a point, a line, or a plane. The new figure is the image and the original figure is the preimage. The preimage and image are congruent, so a reflection is a congruence transformation or isometry.

Example 1:

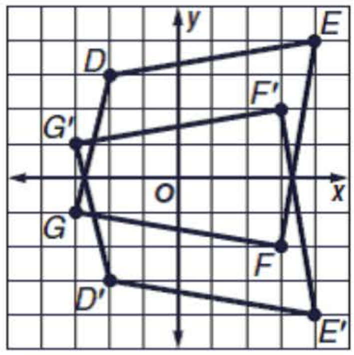
Construct the image of quadrilateral $ABCD$ under a reflection in line m .



Draw a perpendicular from each vertex of the quadrilateral to m. Find vertices A', B', C' and D' that are the same distance from m on the other side of m. The image is $A'B'C'D'$.

Example 2:

Quadrilateral $DEFG$ has vertices $D(2,3), E(4,4), F(3,2)$ and $G(3,1)$. Find the image under reflection in the x-axis.



To find an image for a reflection in the x-axis, use the same x-coordinate and multiply the y-coordinate by -1. In symbols, $(a,b) \longrightarrow (a,-b)$. The new coordinates are $D'(-2,3), E'(4,-4), F'(3,2)$, and $G'(-3,1)$.
The image is $D'E'F'G'$.

In example 2, the notation $(a,b) \longrightarrow (a,-b)$ represents a reflection in the x-axis. Here are three other common reflections in the coordinate plane.

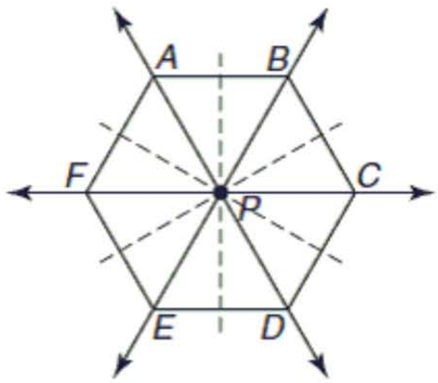
. in the y-axis: $(a,b) \longrightarrow (-a,b)$
. in the line $y = x : (a,b) \longrightarrow (b,a)$
. in the origin: $(a,b) \longrightarrow (-a,-b)$

3.11.2 Lines and Points of Symmetry

If a figure has a line of symemtry, then it can be folded along that line so that the two halves match. If a figure has a point of symmetry, it is the midpoint of all segments between the preimage and image points.

Example
Determine how many lines of symmetry a regular hexagon has. Then determine whether a regular hexagon has point symmetry.

There are six lines of symmetry, three that are diagonals through opposite vertices and three that are perpendicular bisectors of opposite sides. The hexagon has point symmetry because any line through P identifies two points on the hexagon that can be considered images of each other.



3.12 Translation

A translation, or translation operator, is an affine transformation of Euclidean space which moves every point by a fixed distance in a specified direction. It is one of the regid motions (other rigid motions include rotation and reflection). It can also be interpreted as the addition of a constant vector to every point, or as shifting the origin of the coordinate system. In other words, if v is a fixed vector, then the translation Tv will work as $Tv(p) = p + v$.
If T ia a translation, then the image of a subset A under the fuction T is the translate of A by T. The translate of A by Tv is often writen $A + v$.
In Euclidean space, any translation is an isometry. The set of all translations forms the translation group T, which is isomorphic to the space it self, and a normal subgroup of Euclidean group $E(n)$ by T is isomorphic to the orthogonal group $O(n) : E(n)/T \cong O(n)$.

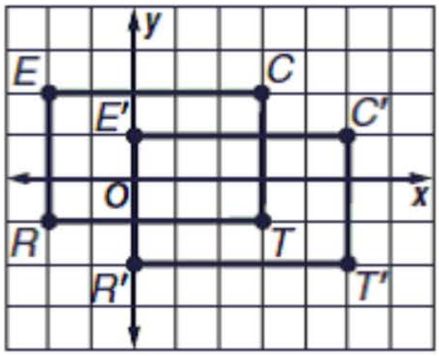
3.12.1 Translation Using Coordinate

A transformation called a translation slides a figure in a given direction. In the coordinate plane, a translation moves every preimage point $P(x, y)$ to an image point $P(x+a, y+b)$ for fixed values a and b. In words, a translation shifts a figure a units horizontally and b units vertically; in symbols, $(x, y) \longrightarrow (x + a, y + b)$.

Example
Rectangle $RECT$ has vertices $R(-2, -1)$, $E(-2, 2)$, $C(3, 2)$ and $T(3, -1)$.

Graph $RECT$ and its image for the translation $(x, y) \longrightarrow (x + 2, y - 1)$.

The translation moves every point of the preimage right 2 units and down 1 unit.
$(x, y) \longrightarrow (x + 2, y - 1)$
$R(-2, -1) \longrightarrow R'(-2 + 2, -1 - 1)$ or $R'(0, -2)$
$E(-2, 2) \longrightarrow E'(-2 + 2, 2 - 1)$ or $E'(0, 1)$
$C(3, 2) \longrightarrow C'(3+2, 2-1)$ or $C'(5, 1)$
$T(3, -1) \longrightarrow T'(3 + 2, -1 - 1)$ or $T'(5, -2)$

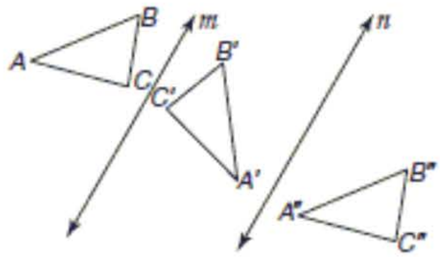


3.12.2 Translation by Repeat Reflections

Another way to find the image of a translation is to reflect the fingure twice in parallel lines. This kind of translation is called a composite of reflections.

Example:

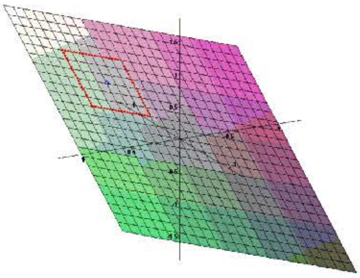
In the fingure, $m \parallel n$. Find the translation image of ΔABC.
$\Delta A'B'C'$ is the image of ΔABC reflected in line m. $\Delta A''B''C''$ is the image of $\Delta A'B'C'$ reflected in line n. The final image, $\Delta A''B''C''$, is a translation of ΔABC.



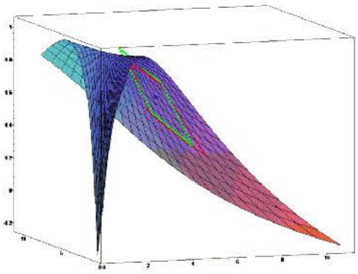
3.13 Generalization

More generally, a transformation in mathematics means a mathematical function (synonyms: map and mapping). A transformation can be an invertible function from a set X to itself, or from X to another set Y. The choice of the term transformation may simply flag that a function's more geometric aspects are being considered (for example, with attention paid to invariants).

A strong non-linear transformation applied to a plane through the origin



Before Transformation



After Transformation

Chapter 4

Pencil of Conics

4.1 Pencil of Lines

Lemma 1:
If $(a, b) \neq (0, 0)$ then the equation $ax + by + c = 0$ represents a line.
Proof:
The case $b \neq 0$: the equation is equivalent to $y = -\frac{a}{b}x - \frac{c}{b}$, which represents a line with slope $-\frac{a}{b}$.
The case $b = 0$: from the assumption $(a, b) \neq (0, 0)$ we see that $a \neq 0$. Therefore the equation is equivalent to $x = -\frac{c}{a}$, which represents a line parallel to y-axis.
Example 1: For any $a, c \in \mathbb{R}$. The equation $ax + (1 - a)y + c = 0$ represents a line.
This is easily shown, but to show how to apply Lemma 1, we give another proof.
Proof:
Suppose $a = 0$ and $1 - a = 0$, $\cdots (\star)$
then $a = 0$ and $a = 1$
then $0 = 1$, contradiction.
Therefore the assumption $(\star)$ is false.
i.e,. $a \neq 0$ or $1 - a \neq 0$, then by the lemma 1, the equation represents a line.
Definition: Let A be a given point in the plane $\mathbb{R}^2$. The set of all the lines passing through A is called the pencil of lines on A, and denoted by $\rangle A \langle$:

$$\rangle A \langle = \{\ell \subset \mathbb{R}^2 |\ \ell \text{ is a line, } \ell \ni A\}$$

Notation: We denote the line with equation $ax + by + c = 0$ by $\ell(a, b, c)$
Example 2: Line $y = 2x + 3$ is denoted by $\ell(2, -1, 3)$ or $\ell(-2, 1, -3)$ or $\ell(1, \frac{1}{2}, \frac{3}{2})$ or $\cdots$, because $y = 2x + 3 \Longleftrightarrow 2x - y + 3 = 0 \Longleftrightarrow -2x + y - 3 = 0 \Longleftrightarrow x - \frac{1}{2}y + \frac{3}{2} = 0 \Longleftrightarrow \cdots$

Lemma:
$\ell(a_1, b_1, c_1) = \ell(a_2, b_2, c_2) \iff (a_1, b_1, c_1)$ is proportional to (a_2, b_2, c_2)
i.e.,$a_1 : b_1 : c_1 = a_2 : b_2 : c_2$ or $a_1b_2 = a_2b_1$ and $b_1c_2 = b_2c_1$ and $a_1c_2 = a_2c_1$
i.e., $a_1b_2 - a_2b_1 = b_1c_2 - b_2c_1 = a_1c_2 - a_2c_1 = 0$
Corollary:
(1)$\ell(a_1, b_1, c_1) \neq \ell(a_2, b_2, c_2) \iff$ at least one of the following is non-zero:

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \quad \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, \quad \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

(2)$\ell(a_1, b_1, c_1)$ and $\ell(a_2, b_2, c_2)$ are not parallel to each other if and only if

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$$

Lemma
Let $\ell_1 : a_1x + b_1y + c_1 = 0$ and $\ell_2 : a_2x + b_2y + c_2 = 0$
if ℓ_1 and ℓ_2 are not parallel to each other, then for any $(s, t) \neq (0, 0)$ the linear combination

$$s(a_1x + b_1y + c_1) + t(a_1x + a_2y + c_2) = 0$$

represents a line passes through the intersection point of ℓ_1 and ℓ_2.
Proof: The equation equivalent to

$$(sa_1 + ta_2)x + (sb_1 + tb_2)y + (sc_1 + tc_2) = 0$$

By Lemma 1, if $\underline{sa_1 + ta_2 \neq 0 \text{ or } sb_1 + tb_2 \neq 0}$, then the equation represents a line. We proof __ by contraditction.
Suppose

$$\begin{cases} sa_1 + ta_2 &= 0 \\ sb_1 + tb_2 &= 0 \end{cases} \quad i.e,. \quad \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

since $\ell_1 \nparallel \ell_2$ then
$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1 \neq 0$ by corollary (2)
then A^{-1} exists, and we have
$\begin{pmatrix} s \\ t \end{pmatrix} = A^{-1} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
$\begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
This contradicts to the assumption $\begin{pmatrix} s \\ t \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Therefore, the equation $(sa_1 + ta_2)x + (sb_1 + tb_2)y + (sc_1 + tc_2) = 0$ represents a line pass through ℓ_1 and ℓ_2.

Lemma:

Given a point $A(x_A, y_A)$ and two line $\ell_i : a_ix + b_iy + c_i = 0 \quad (i = 1, 2)$ passing through A.

(1) For each pair $s, t \in \mathbb{R}$ with $(s, t) \neq (0, 0)$,
define a line $\ell_{s,t} : s(a_1x + b_1y + c_1) + t(a_2x + b_2y + c_2) = 0$.
Then for any s, t the line $\ell_{s,t}$ pass through A.

(2)If a line ℓ passes through A then $\ell = \ell_{s,t}$ for some s, t with $(s, t) \neq (0, 0)$.

Proof:

Since $\ell_i = a_ix + b_iy + c_i = 0$ passes through A, we have

$$a_1x_A + b_1y_A + c_1 = 0, \qquad a_2x_A + b_2y_A + c_2 = 0 \qquad\qquad (\star)$$

(1) From $(\star)$, we have

$$s(a_1x_A + b_1y_A + c_1) + t(a_2x_A + b_2y_A + c_2) = 0.$$

This mean that the line $\ell_{s,t}$ pass through A.

(2) Let ℓ be a line passing through A. We must find $(s, t) \neq (0, 0)$ such that the equation

$$s(a_1x + b_1y + c_1) + t(a_2x + b_2y + c_2) = 0 \cdots (\star\star)$$

represents ℓ.

Chose $B(x_B, y_B) \in \ell$ different from A.

To find (s, t) such that $\ell = \ell_{s,t}$ with $(s, t) \neq (0, 0)$, substitute the coordinate of $B(x_B, y_B)$ to $(\star\star)$, then we have

$$s(a_1x_B + b_1y_B + c_1) + t(a_2x_B + b_2y_B + c_2) = 0.$$

In order to get s, t satisfying above, we choose $s = a_2x_B+b_2y_B+c_2$, $t = -(a_1x_B+b_1y_B + c_1)$. We must show $(s, t) \neq (0, 0)$. We prove this by contradiction. Suppose $(s, t) = (0, 0)$

$$\begin{cases} a_1x_B + b_1y_B + c_1 &= 0 \\ a_2x_B + b_2y_B + c_2 &= 0 \end{cases}$$

The equations above implies that $B \in \ell_1 \cap \ell_2$.

On the other hand, $A \in \ell_1 \cap \ell_2$.

Since $\ell_1 \nparallel \ell_2$, $\ell_1 \cap \ell_2$ consists of single point. Therefore $A = B$. This contradicts to the assumption that we have chosen B different from A.

Therefore, $(s,t) \neq (0,0)$. So the equation $(sa_1+ta_2)x+(sb_1+tb_2)y+(sc_1+tc_2) = 0$ represents a line $\ell_{s,t}$. $\ell_{s,t}$ passes through A since

$$s(a_1x_A + b_1y_A + c_1) + t(a_2x_A + b_2y_A + c_2) = 0.$$

and $\ell_{s,t}$ passes through B since

$$s(a_1x_B + b_1y_B + c_1) + t(a_2x_B + b_2y_B + c_2) = 0.$$

ℓ passing through A and B also. Since B different from A, we have $\ell_{s,t} = \ell$.
Example 3: Let $A = (1,2)$, and let $\ell_1 : 2x + 3y - 8 = 0, \quad \ell_2 : 2x - 3y + 4 = 0$. We see that ℓ_1 and ℓ_2 pass through A, i.e., $\ell_1, \ell_2 \in \rangle A \langle$, and that $\ell_1 \neq \ell_2$. Then, by the above theorem, we have

$$\begin{aligned} \rangle A \langle &= \{\ell(2s+2t, 3s-3t, -8s+4t) | (s,t) \in \mathbb{R} \text{ and } (\mathrm{s,t}) \neq (0,0)\} \\ &= \{\ell(2s+2t, 3s-3t, -8s+4t) | s = \cos\theta, t = \sin\theta, \text{for some } \theta \in \mathbb{R}\} \end{aligned}$$

Example 4: Given two lines $\ell_1 : x + y - 4 = 0$ and $\ell_2 : 2x + 3y - 6 = 0$. Let A be the intersection point of ℓ_1 and ℓ_2. Find the equation of the line ℓ passing through A and $B(3,2)$.
Answer (1):
Let $\ell : ax + by + c = 0$
By solving the system equation

$$\begin{cases} x + y - 4 &= 0 \\ 2x + 3y - 6 &= 0 \end{cases}$$

we obtain $A(6,-2)$.
ℓ passing through $A(6,-2)$, therefore

$$6a - 2b + c = 0 \cdots\cdots (1)$$

ℓ passing through $B(3,2)$, therefore

$$3a + 2b + c = 0 \cdots\cdots (2)$$

By (1)+ (2), then

$$9a + 2c = 0$$

$$a = -\frac{2c}{9} \cdots\cdots (3)$$

Substitute (3) to (1), we obtain $b = -\dfrac{c}{6} \cdots\cdots (4)$
For example, we choose $c = 18$ then $a = -4, \quad b = -3$
Therefore the equation of ℓ is

$$-4x - 3y + 18 = 0.$$

Answer(2): By the previous lemma(2), the equation of ℓ can be obtained as lininear combination of equations of ℓ_1 and ℓ_2:

$$\ell : s(x + y - 4) + t(2x + 3y - 6) = 0 \cdots\cdots (\star)$$

Since ℓ passes through $B(3, 2)$, we have

$$\begin{aligned} s(3 + 2 - 4) + t(6 + 6 - 6) &= 0 \\ s + 6t &= 0 \end{aligned}$$

For example we choosee $t = -1$ and $s = 6$ then $(\star)$ is equivalent to

$$\begin{aligned} 6(x + y - 4) - (2x + 3y - 6) &= 0 \\ 4x + 3y - 18 &= 0 \end{aligned}$$

Therefore the equation of the line ℓ passing through A and B is $-4x - 3y + 18 = 0$.

Theorem:
Let A : be a given point. If

$$\ell(a_1, b_1, c_1), \ell(a_2, b_2, c_2) \in \rangle A\langle, \quad \ell(a_1, b_1, c_1) \neq \ell(a_2, b_2, c_2)$$

then

$$\begin{aligned} \rangle A\langle &= \{\ell(sa_1 + ta_2, sb_1 + tb_2, sc_1 + tc_2) | (s, t) \in \mathbb{R} \text{ and } (\mathrm{s}, \mathrm{t}) \neq (0, 0)\} \\ &= \{\ell(sa_1 + ta_2, sb_1 + tb_2, sc_1 + tc_2) | s = \cos\theta, t = \sin\theta, \text{for some } \theta \in \mathbb{R}\} \end{aligned}$$

Thus, the set $\rangle A\langle$ is described in term of one parameter θ. i.e,. dimension of the set $\rangle A\langle$ is one.

4.2 Pencil of Circles

Notation: We denote the circle with center (a, b) and radius r by $C(a, b, r)$.
Example1: Given a point $A(x_A, y_A)$ in plan, let $\mathcal{S}$ be the set of all the circles $\ni A$. The dimension of this set is two, because $\mathcal{S} = \{C(a, b, r)| \ r = \sqrt{(a - x_A)^2 + (b - y_A)^2}, \ \ a, b \in \mathbb{R}^2\}$, and is described in term of two parameters a and b.
Example2: Given two distinct points $A = (k, 0)$, $B = (-k, 0)$ in plan. Let $\mathcal{S}$ be the set of all circles $\ni A, B$. The dimension of this set is one, because $\mathcal{S} = \{C(0, b, \sqrt{1 + b^2} \mid b \in \mathbb{R})\}$, and is described in term of one parameter b.
Example3: Given three distinct points A, B, C, the set of all the circles $\ni A, B, C$ has dimension zero.

Among these three cases, the case of dimension one is most interesting.

Example4: Given two circles $C_1 : x^2 + y^2 = 4$ and $C_2 : (x-2)^2 + y^2 = 1$. Let A, B be the intersection points of C_1 and C_2. Suppose the circle C passes through A and B.

(1) Suppose $C \ni (3, 2)$. Find the equation of C.

(2) Suppose the center of the circle C is (3,0). Find the equation of C.

Answer:

(1) Find the equation of C.

Suppose the equation of C is $(x-a)^2 + (y-b)^2 = r^2$.

The points $C \ni (3, 2)$.

To find the intersection of C_1 and C_2, we solve the system of quadratic equations:

$$\begin{cases} x^2 + y^2 - 4 &= 0 \quad \cdots\cdots(1) \\ x^2 + y^2 - 4x + 3 &= 0 \quad \cdots\cdots(2) \end{cases}$$

By (1)−(2), we obtain

$$\begin{aligned} 4x - 7 &= 0 \\ x &= \frac{7}{4} \quad \cdots\cdots(3) \end{aligned}$$

substitute (3) to (1) then we obtain $y = \pm\frac{\sqrt{15}}{4}$.

The circle C pass through C_1, C_2 at the points $(3, 2)$

$$\Longleftrightarrow \begin{cases} (3-a)^2 + (2-b)^2 &= r^2 \quad \cdots\cdots(4) \\ \left(\frac{7}{4} - a\right)^2 + \left(\frac{\sqrt{15}}{4} - b\right)^2 &= r^2 \quad \cdots\cdots(5) \\ \left(\frac{7}{4} - a\right)^2 + \left(-\frac{\sqrt{15}}{4} - b\right)^2 &= r^2 \quad \cdots\cdots(6) \end{cases}$$

by (5)−(6), we obtain

$$\begin{aligned} \left(\frac{\sqrt{15}}{4} - b\right)^2 - \left(\frac{\sqrt{15}}{4} + b\right)^2 &= 0 \\ -4 \times \frac{\sqrt{15}}{4} b &= 0 \\ b &= 0 \quad \cdots\cdots(7) \end{aligned}$$

substitute (7) to (4) we obtain $(3-a)^2+4=r^2 \quad \cdots\cdots(8)$
we solve the system of quadratic equations of (7) and (8):

$$\begin{cases} (3-a)^2+4 &= r^2 \quad \cdots\cdots(8) \\ \left(\frac{7}{4}-a\right)^2+\frac{15}{16} &= r^2 \quad \cdots\cdots(9) \end{cases}$$

By (8)−(9), we obtain

$$\begin{aligned} (9-6a+a^2+4-r^2)-\left(\frac{49}{16}-\frac{7}{2}a+a^2+\frac{15}{16}-r^2\right) &= 0 \\ -\frac{5}{2}a &= -\frac{144}{16} \\ a &= \frac{18}{5} \quad \cdots\cdots(10) \end{aligned}$$

Substitute (10) to (4), we obtain

$$\begin{aligned} \left(3-\frac{18}{5}\right)^2+4 &= r^2 \\ r^2 &= \frac{109}{25} \end{aligned}$$

Therefore the equation of C is: $\left(x-\frac{18}{5}\right)^2+y^2-\frac{109}{25}=0.$

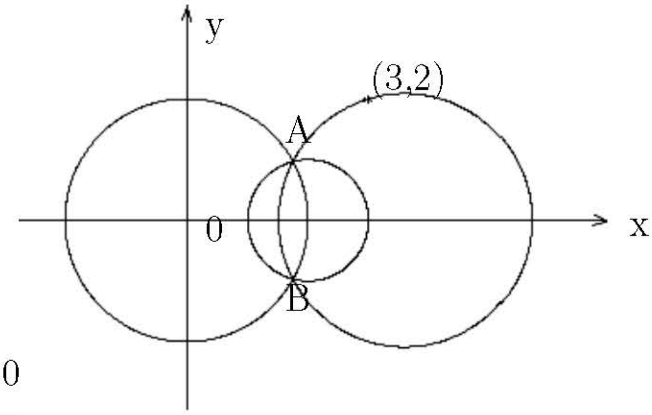
Another way:

By analogy to the case of pencil of line (see Exercise 4) the equation of C will be obtained as linear combination of the equations of C_1 and C_2.

$C : s(x^2 + y^2 - 4) + t(x^2 + y^2 - 4x + 3) = 0$ *for some* $s, t \in \mathbb{R}$
i.e,. $(s + t)x^2 + (s + t)y^2 - 4tx - 4s + 3t = 0 \quad \cdots\cdots \ (\star)$

The points
$C \ni (3, 2) \iff 9(s + t) + 4(s + t) - 12t - 4s + 3t = 0$

$$9s + 4t = 0.$$



For example we choose $s = -4$ and $t = 9$ then $(\star)$ is equivalent to

$$(-4 + 9)x^2 + (-4 + 9)y^2 - 36x + 43 = 0$$
$$5x^2 + 5y^2 - 36x + 43 = 0$$
$$x^2 + y^2 - \frac{36}{5}x + \frac{43}{5} = 0$$

Therefore the equation of C is: $\left(x - \frac{18}{5}\right)^2 + y^2 - \frac{109}{25} = 0.$

(2) Denote the radus of C by r
Then the equation of C is $(x - 3)^2 + y^2 = r^2 \cdots\cdots (\star)$.
In order to find the intersection of C_1 and C_2, we solve the system of equation:

$$\begin{cases} x^2 + y^2 - 4 &= 0 \cdots\cdots (1) \\ x^2 + y^2 - 4x + 3 &= 0 \cdots\cdots (2) \end{cases}$$

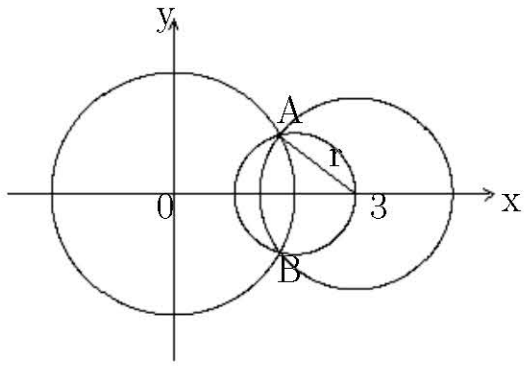
by (1)-(2), we have

$$4x - 7 = 0$$
$$x = \frac{7}{4} \cdots\cdots (3)$$

substitute (3) to (1) then we have

$$\left(\frac{7}{4}\right)^2 + y^2 - 4 = 0$$

$$y = \pm\frac{\sqrt{15}}{4} \quad \cdots\cdots (4)$$



substitute (3) and (4) to ($\star$), then we obtain $r^2 = \left(\frac{7}{4} - 3\right)^2 + \frac{15}{6} = \frac{5}{2}$

Therefore ($\star$) $\iff (x-3)^2 + y^2 - \frac{5}{2} = 0$

i.e,. $x^2 + y^2 - 6x + \frac{13}{2} = 0$

$\iff 2x^2 + 2y^2 - 12x + 13 = 0.$

Therefore the equation of C is: $2x^2 + 2y^2 - 12x + 13 = 0.$

Another way:

By analogy to the case of pencil of line (see Exercise 4) the equation of C will be obtained as linear combination of the equations of C_1 and C_2.

$C : s(x^2 + y^2 - 4) + t(x^2 + y^2 - 4x + 3) = 0$ for some $s, t \in \mathbb{R}$

i.e,. $(s+t)x^2 + (s+t)y^2 - 4tx - 4s + 3t = 0 \quad \cdots\cdots (\star)$

$$\begin{aligned} x^2 + y^2 - \frac{4t}{s+t}x - \frac{4s-3t}{s+t} &= 0 \\ \left(x - \frac{2t}{s+t}\right)^2 - \frac{4t^2}{(s+t)^2} + y^2 - \frac{4s-3t}{s+t} &= 0 \end{aligned}$$

it represents the circle with the center $\left(\frac{2t}{s+t}, 0\right)$.

The circle C with the center $(3, 0)$, then

$$\begin{aligned} 2t &= 3(s+t) \\ -t &= 3s \end{aligned}$$

example s=1 and t= -3 then ($\star$) is equivalent to $-2x^2 - 2y^2 + 12x - 13 = 0.$

Therefore the equation C is: $2x^2 + 2y^2 - 12x + 13 = 0.$

Theorem:

Given two distinct points A and B. Choose two distinct circles C_1, C_2 both passing through A and B with equations $C_1 : x^2 + y^2 - 2a_1x - 2b_1y + c_1 = 0$ and $C_2 : x^2 + y^2 - 2a_2x - 2b_2y + c_2 = 0$. Then, for any circle C passing through

A and B, its equation is obtained as a linear combination of the equations of C_1 and C_2, i.e,. C is represented by the following equation:

$$C : (s+t)x^2 + (s+t)y^2 - (2a_1 s + 2a_2 t)x - (2b_1 s + 2b_2)y + (c_1 s + c_2 t) = 0$$

for some $(s,t) \neq (0,0)$. Moreover,we can choose $s = \cos\theta,\ t = \sin\theta$ for some $\theta \in \mathbb{R}$.

Proof:

Let C be a circle passes through A and B. By the above theorem, C can be represented by

$$s(x^2 + y^2 - 2a_1 x - 2b_1 y + c_1) + t(x^2 + y^2 - 2a_2 x - 2b_2 y + c_2) = 0 \qquad (\star)$$

for some $(s,t) \neq (0,0)$. We must find such $(s,t) \neq (0,0)$.
Choose $D(x_D, y_D) \in C$ different from A and B then
In order to find $(s,t) \neq (0,0)$, substitute the coordinate of $D(x_D, y_D)$ to $(\star)$, then we have

$$s(x_D^2 + y_D^2 - 2a_1 x_D - 2b_1 y_D + c_1) + t(x_D^2 + y_D^2 - 2a_2 x_D - 2b_2 y_D + c_2) = 0$$

In order to get (s,t) satisfying above, we choose

$$\begin{aligned} s &= x_D^2 + y_D^2 - 2a_2 x_D - 2b_2 y_D + c_2 \\ t &= -(x_D^2 + y_D^2 - 2a_1 x_D - 2b_1 y_D + c_1) \end{aligned}$$

then we must show $(s,t) \neq (0,0)$. We prove this by comtradiction.
Suppose $(s,t) = (0,0)$, then

$$\begin{cases} x_D^2 + y_D^2 - 2a_2 x_D - 2b_2 y_D + c_2 = 0 \\ x_D^2 + y_D^2 - 2a_1 x_D - 2b_1 y_D + c_1 = 0 \end{cases}$$

The equations above imply that $D \in A \cap B$.
On the other hand, $C_1 \cap C_2 = \{A, B\}$.
So we obtain $D = A$ or $D = B$.
This is contradiction that we have choosen D different from A and B.
Therefore, $(s,t) \neq (0,0)$. So the equation

$$s(x^2 + y^2 - 2a_1 x - 2b_1 y + c_1) + t(x^2 + y^2 - 2a_2 x - 2b_2 y + c_2) = 0$$

represents a circle passing through A and B.
And the circle passes through D since

$$s(x_D^2 + y_D^2 - 2a_1 x_D - 2b_1 y_D + c_1) + t(x_D^2 + y_D^2 - 2a_2 x_D - 2b_2 y_D + c_2) = 0.$$

On the other hand, the circle C passes though A, B and D also.
Since A, B and D are distinct, the circle coincides with C.

4.3 Pencil of Conics:

In previous sections we have defined *pencil of lines* and *pencil of circles*. In analogy to these, we define *pencil of conics*.

Definition:

Given two conics C_1 and C_2 with equations:

$$C_1 : a_1x^2 + b_1xy + c_1y^2 + d_1x + e_1y + f_1 = 0$$

and

$$C_2 : a_2x^2 + b_2xy + c_2y^2 + d_2x + e_2y + f_2 = 0.$$

The set of all conics represented by

$$s(a_1x^2 + b_1xy + c_1y^2 + d_1x + e_1y + f_1) + t(a_2x^2 + b_2xy + c_2y^2 + d_2x + e_2y + f_2) = 0$$

for some $(s,t) \neq (0,0)$ is called *pencil of conics*.

In other words, *pencil of conics* is the set of all conics represented by $ax^2 + bxy + cy^2 + dx + ey + f = 0$ with

$$a = a_1s + a_2t,\ b = b_1s + b_2t,\ c = c_1s + c_2t,\ d = d_1s + d_2t,\ e = e_1s + e_2t,\ f = f_1s + f_2t$$

for some $(s,t) \neq (0,0)$. We can choose $s = \cos\theta, \quad t = \sin\theta$ for some $\theta \in \mathbb{R}$.

Example1: Given two conics $C_1 : x^2 + y^2 = 5$ and $C_2 : x^2 - y^2 = 3$. Find all the conics which pass through all the intersection points of C_1 and C_2.

Answer:

By solving the system of equations $x^2 + y^2 = 5$ and $x^2 - y^2 = 3$, we get four intersection points $A(2,1), B(2,-1), C(-2,1), D(-2,-1)$. Let $\mathcal{C}$ be a conic which passes through A, B, C, D. By analogy to the case of pencil of lines (see Execise 4), the equation of $\mathcal{C}$ will be obtain by linear combination of the equations of C_1 and C_2:

$C :\ s(x^2 + y^2 - 5) + t(x^2 - y^2 - 3) = 0$ for some $s, t \in \mathbb{R}$.

$$i.e.,\quad (s+t)x^2 + (s-t)y^2 - (5s+3t) = 0 \cdots (\star)$$

We prove this.

Let $\mathcal{C} : ax^2 + bxy + cy^2 + dx + ey + f = 0$.

$\mathcal{C} \ni A(2,1)$ imply

$$4a + 2b + c + 2d + e + f = 0 \cdots (1)$$

$\mathcal{C} \ni B(2,-1)$ imply

$$4a - 2b + c + 2d - e + f = 0 \cdots (2)$$

$\mathcal{C} \ni C(-2,1)$ imply

$$4a - 2b + c - 2d + e + f = 0 \cdots (3)$$

$\mathcal{C} \ni D(-2,-1)$ imply

$$4a + 2b + c - 2d - e + f = 0 \cdots (4)$$

By solving these system equation, we obtain

$$\begin{aligned} 4a + c + f &= 0 \\ b &= 0 \\ d &= 0 \\ e &= 0 \end{aligned}$$

We choose a and c are arbitrary then $f = -4a - c$
Therefore the equation of $\mathcal{C}$ is $ax^2 + cy^2 - 4a - c = 0$.
Let $s = \dfrac{a+c}{2}$ and $t = \dfrac{a-c}{2}$. Then we obtain $s+t = a$, $s-t = c$, and moreover $5s + 3t = 4a + c$.
Therefore the conic $\mathcal{C}$ is represented by the equation ($\star$), as is expected.

Example2: Given two conics $C_1 : x^2 - y^2 = 1$ and $C_2 : y^2 - x - 5 = 0$.
Find all the conics which pass through all the intersection points of C_1 and C_2.

Answer:
By analogies to the case of pencil of lines (see Execise 4). The equation of C will be obtain by linear combination of the equations of C_1 and C_2:
$C: \; s(x^2 - y^2 - 1) + t(y^2 - x - 5) = 0$ for some $s, t \in \mathbb{R}$.

$$i.e., \quad sx^2 - (s-t)y^2 - tx - (s+5t) = 0 \cdots \cdot (\star)$$

We prove this.
By solving the system of equations $x^2 - y^2 - 1 = 0$ and $y^2 - x - 5 = 0$, we get four intersection points $A(3, \sqrt{8}), B(3, -\sqrt{8}), C(-2, \sqrt{3}), D(-2, -\sqrt{3})$.
Let $\mathcal{C} : ax^2 + bxy + cy^2 + dx + ey + f = 0$.
$\mathcal{C} \ni A(3, \sqrt{8})$ imply

$$9a + 3\sqrt{8}b + 8c + 3d + \sqrt{8}e + f = 0 \cdots (1)$$

$\mathcal{C} \ni B(3, -\sqrt{8})$ imply

$$9a - 3\sqrt{8}b + 8c + 3d - \sqrt{8}e + f = 0 \cdots (2)$$

$\mathcal{C} \ni C(-2, \sqrt{3})$ imply

$$4a - 2\sqrt{3}b + 3c - 2d + \sqrt{3}e + f = 0 \cdots (3)$$

$\mathcal{C} \ni D(-2, -\sqrt{3})$ imply

$$4a + 2\sqrt{3}b + 3c - 2d - \sqrt{3}e + f = 0 \cdots (4)$$

By solving these system equation, we obtain

$$\begin{aligned} a - 5d + f &= 0 \\ c + 6d - f &= 0 \\ e &= 0 \end{aligned}$$

We choose d and f are arbitrary then

$$\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} = \begin{pmatrix} 5dd - f \\ -6d + f \\ d \\ 0 \\ f \end{pmatrix}$$

Therefore the equation of $\mathcal{C}$ is $(5d - f)x^2 - (6d - f)y^2 + dx + f = 0$, for some $(s, t) \neq (0, 0)$
Let $s = 5d - f$ and $t = -d$.
Then we obtain $6d - f = s - t$, $f = -(5t + s)$.
Therefore the conic $\mathcal{C}$ is represented by the equation $(\star)$, as is expected.

Example3: Given two conics $C_1 : x^2 + y^2 = 1$ and $C_2 : x^2 - y^2 = 1$. Find all the conics which pass through all the intersection points of C_1 and C_2.

Answer :
Can we expect, by analogy to the case of pencil of lines (see Execise 4), that the equation of will be obtain by linear combination of the equations of C_1 and C_2 ?
$s(x^2 + y^2 - 1) + t(x^2 - y^2 - 1) = 0$ for some $s, t \in \mathbb{R}$

$$i.e., \quad (s + t)x^2 + (s - t)y^2 - (s + t) = 0. \cdots\cdot (\star)$$

By solving the system of equations $x^2+y^2-1=0$ and $x^2-y^2-1=0$, we get only two intersection points $A(1,0), B(-1,0)$. For example, parabola $x^2-y-1=0$ passes through A and B. This can not be obtained as $(\star)$ because this has linear term but $(\star)$ has no linear term.
So the problem can not be solved in this way.
We need to add some conditions to sovle this problem.

Example3$'$: Given two conics $C_1 : x^2+y^2=1$ and $C_2 : x^2-y^2=1$.
They meet at two points and at each intersection point, they are tangent to each other.
Find all the conics which pass through all the intersection points of C_1 and C_2 and which are tangent to C_1 and C_2 at the intersection points.

Answer:
By analogies to the case of pencil of lines (see Execise 4). The equation of C will be obtain by linear combination of the equations of C_1 and C_2:
$C:\ s(x^2+y^2-1)+t(x^2-y^2-1)=0$ for some $s,t\in\mathbb{R}$.

$$i.e.,\quad (s+t)x^2+(s-t)y^2-(s+t)=0\cdots\cdot(\star)$$

We prove this.
We know the intersection points are $A(1,0)$ and $B(-1,0)$.
Let $\mathcal{C} : ax^2+bxy+cy^2+dx+ey+f=0\cdots(\star\star)$, where $(a,b,c)\neq(0,0,0)$
$\mathcal{C}\ni A(1,0)$ imply

$$a+d+f=0\cdots(1)$$

$\mathcal{C}\ni B(-1,0)$ imply

$$a-d+f=0\cdots(2)$$

we need to find equation of the line ℓ tangent to C at $A(1,0)$.
Let $\ell : x=ky+1$
Substitute $x=ky+1$ to the $(\star\star)$, we obtain

$$\begin{aligned} a(ky+1)^2+b(ky+1)y+cy^2+d(ky+1)+ey+f&=0\\ (ak^2+bk+c)y^2+(2ak+b+dk+e)y+(a+d+f)&=0\\ By(1),(ak^2+bk+c)y^2+(2ak+b+dk+e)y+0&=0 \end{aligned}$$

Let Δ be discriminant of the equation above, then $\Delta=(2ak+b+dk+e)^2$ The line ℓ tangents to C if and only if $\Delta = 0$. So we obtain

$$\begin{aligned} 2ak + b + dk + e &= 0 \\ k &= \frac{-e-b}{2a+d} \end{aligned}$$

The tangent line to C at A is equal to the tangent line to C_1 at A, which is $x = 1$. So $k = 0$, and we have $b + e = 0 \cdots\cdots (3)$.
We need to find equation of the line tangent at $B(-1, 0)$
Let $\ell : x = ky + 1$
The line ℓ tangent to C.
Substitute $x = ky + 1$ to C, we obtain

$$\begin{aligned} a(ky+1)^2 - b(ky+1)y + cy^2 - d(ky+1) + ey + f &= 0 \\ (ak^2 - bk + c)y^2 + (2ak - b - dk + e)y + (a - d + f) &= 0 \\ \Delta &= (2ak + b + dk + e)^2 \end{aligned}$$

The line ℓ tangent to C at B if and only if $\Delta = 0$, then

$$\begin{aligned} 2ak - b - dk + e &= 0 \\ k &= \frac{-b+e}{2a-d} \end{aligned}$$

The tangent line to C at A is equal to the tangent line to C_1 at A, which is $x = -1$. So $k = 0$ then $-b + e = 0 \cdots\cdots (4)$
By solving these system equation, we obtain

$$\begin{aligned} a + f &= 0 \\ b &= 0 \\ d &= 0 \\ e &= 0 \end{aligned}$$

We choose d and f are arbitrary then

$$\begin{pmatrix} a \\ b \\ d \\ e \\ f \end{pmatrix} = \begin{pmatrix} -f \\ 0 \\ 0 \\ 0 \\ f \end{pmatrix}$$

Therefore the equation of $\mathcal{C}$ is $-fx^2 + cy^2 + f = 0$.
We want to show the equation $(\star)$ represents C for some $(s,t) \neq (0,0)$.
Let $s = \dfrac{-f+c}{2}$ and $t = \dfrac{-f-c}{2}$
Then we obtain $s + t = -f$, $s - t = c$ and $-(s+t) = f$.
Suppose $(s,t) = (0,0)$ then $f = 0$, $c = 0$, which imply $(a,b,c) = (0,0,0)$. This contradicts to the fact that $(\star\star)$ is quadratic equation. So $(s,t) \neq (0,0)$.
Therefore the equation $(\star)$: $(s+t)x^2 + (s-t)y^2 - (s+t) = 0$ represents the conic $\mathcal{C}$.

Example4:
Given quadratic curve $C : ax^2 + bxy + cy^2 + dx + ey + f = 0$.
Let $A_1(x_1, y_1)$ and $A_2(x_2, y_2)$ such that $A \neq B$.
Let a, b, c, d, e, f are unknown, then the quadratic curve

$$C \ni A_1 \Longleftrightarrow ax_1^2 + bx_1y_1 + cy_1^2 + dx_1 + ey_1 + f = 0$$
$$C \ni A_2 \Longleftrightarrow ax_2^2 + bx_2y_2 + cy_2^2 + dx_2 + ey_2 + f = 0$$

To determine the ratio $a : b : c : d : e : f$, we need five points, then the set of all quadratic curves $\ni A_1, A_2, A_3, A_4$ have dimension one.

Notation: Given two points $A(x_A, y_A)$, $B(x_B, y_B)$ such that $A \neq B$. As is well known, the equation of the line AB is given by

$$\begin{vmatrix} x - x_A & y - y_A \\ x_B - x_A & y_B - y_A \end{vmatrix} = 0 \quad or \quad (y_B - y_A)(x - x_A) - (x_B - x_A)(y - y_A) = 0.$$

For the convenience of later use, we denote the left hand side by $f_{AB}(x,y)$, i.e.,

$$f_{AB}(x,y) = \begin{vmatrix} x - x_A & y - y_A \\ x_B - x_A & y_B - y_A \end{vmatrix} = (y_B - y_A)x - (x_B - x_A)y - x_Ay_B + x_By_A.$$

Lemma:
If A and B are different, the equation $f_{AB}(x,y) = 0$ represents the line AB.
Definition:
Four or more points are said to be in ***general position*** if no three of them are collinear.

Lemma:
Given four points A, B, C, D in plane which are in general position.
1)- The equation

$$f_{AB}(x,y) \cdot f_{CD}(x,y) = 0$$

repesents two crossing lines $AB \cup CD$, and

$$f_{AD}(x,y) \cdot f_{BC}(x,y) = 0$$

represents two crossing lines $AD \cup BC$.
2)- For any $s, t \in \mathbb{R}$ such that $(s,t) \neq (0,0)$, the equation

$$sf_{AB}(x,y) \cdot f_{CD}(x,y) + tf_{AD}(x,y) \cdot f_{BC}(x,y) = 0.$$

represents some quadratic curves passing through points A, B, C, D.

Proof: 1)- The equation

$$f_{AB}(x,y) \cdot f_{CD}(x,y) = 0$$

is equivalent to

$$f_{AB}(x,y) = 0 \quad or \quad f_{CD}(x,y) = 0 \qquad (\star).$$

The first one represent the line AB, the second the line CD.
2)- By the definition of the function f_{AB}, we obtain

$$f_{AB}(x_A, y_A) = \begin{vmatrix} x_A - x_A & y_A - y_A \\ x_B - x_A & y_B - y_A \end{vmatrix} = 0$$
$$f_{AB}(x_B, y_B) = \begin{vmatrix} x_B - x_A & y_B - y_A \\ x_B - x_A & y_B - y_A \end{vmatrix} = 0$$

The above two equations show that $f_{PQ}(P) = 0$, $f(PQ)(Q) = 0$ holds in general.
Therefore

$$\begin{aligned} sf_{AB}(A) \cdot f_{CD}(A) + tf_{AD}(A) \cdot f_{BC}(A) &= s \cdot 0 \cdot f_{CD}(A) + t \cdot 0 \cdot f_{BC}(A) = 0 \\ sf_{AB}(B) \cdot f_{CD}(B) + tf_{AD}(B) \cdot f_{BC}(B) &= s \cdot 0 \cdot f_{CD}(B) + tf_{AD}(B) \cdot 0 \ = 0 \\ sf_{AB}(C) \cdot f_{CD}(C) + tf_{AD}(C) \cdot f_{BC}(C) &= sf_{AB}(C) \cdot 0 + tf_{AD}(C) \cdot 0 \quad = 0 \\ sf_{AB}(D) \cdot f_{CD}(D) + tf_{AD}(D) \cdot f_{BC}(D) &= sf_{AB}(D) \cdot 0 + t \cdot 0 \cdot f_{BC}(D) \ = 0 \end{aligned}$$

Therefore the figure represented by $(\star)$ contains A, B, C, D.

Since $f_{AB}(x,y)$ is of degree 1 and $f_{CD}(x,y)$ is of degree 1, the product of $f_{AB}(x,y).f_{CD}(x,y)$ is of degree 2.
Similarly, $f_{AB}(x,y).f_{CD}(x,y)$ is of degree 2.
The equation $(\star)$ is obtained by linear combinattion of them.

Therefore $(\star)$ is of degree at most 2.
We want to show that the degree of $(\star)$ is exactly 2.

Suppose degree of $(\star)$ is 1, then $(\star)$ represents one line. On the other hand, we have already shown that the figure represnted by $(\star)$ contains the four points A, B, C, D in general positions.
It is contradiction.
Suppose degree of $(\star)$ is 0, then $(\star)$ has only constant term. When the constant is non-zero, $(\star)$ represents empty set, which is imposible because at least four points satisfy $(\star)$.
When the constant is 0, $(\star)$ represents the whole plane, i.e., $(\star)$ holds for all (x, y).
By the assumption $(s, t) \neq (0, 0)$, $(\star)$ is equivalent to:

$$s f_{AB}(x, y) \cdot f_{CD}(x, y) = -t f_{AD}(x, y) f_{BC}(x, y)$$

If $s \neq 0$, then $f_{AB}(x, y) \cdot f_{CD}(x, y) = -\frac{t}{s} f_{AD}(x, y) \cdot f_{BC}(x, y)$
Then $f_{AD}(x, y) \cdot f_{BC}(x, y) = 0$ implies $f_{AB}(x, y) \cdot f_{CD}(x, y) = 0$.
That is, the figure represented by $f_{AD}(x, y) \cdot f_{BC}(x, y) = 0$ is contained in the figure represented by $f_{AB}(x, y) \cdot f_{CD}(x, y) = 0$.
But, by (1),
$f_{AB}(x, y) \cdot f_{CD}(x, y) = 0$ represents two crossing lines $AB \cup CD$,
and $f_{AD}(x, y) \cdot f_{BC}(x, y) = 0$ represents another two crossing lines $AD \cup BC$.
So $AD \cup BC \subseteq AB \cup CD$, it is contradiction.
Thus, we conclude the degree of $(\star)$ is exactly 2.

Example5: Given four points $A(1, 1)$, $B(-1, 1)$, $C(-1, -1)$, $D(1, -1)$, then by the difinition of line pass through two points, we obtain

$$\begin{aligned} f_{AB}(x, y) &= (0x + 1y - 1) \\ f_{CD}(x, y) &= (0x + 1y + 1) \\ f_{AD}(x, y) &= (1x + 0y - 1) \\ f_{BC}(x, y) &= (1x + 0y - 1) \end{aligned}$$

then by the linear combination of lines pass through two points, we obtain

$$\begin{aligned} sf_{AB}(x,y)f_{CD}(x,y) + tf_{AD}(x,y)f_{BC}(x,y) &= 0 \\ s(y-1)(y+1) + t(x-1)(x+1) &= 0 \\ s(y^2-1) + t(x^2-1) &= 0 \\ tx^2 + sy^2 - (s+t) &= 0 \\ tx^2 + sy^2 &= s+t \qquad (\star) \end{aligned}$$

. Case: $s + t \neq 0$ then $(\star)$ is equivalent to $\dfrac{t}{s+t}x^2 + \dfrac{s}{s+t}y^2 = 1$.

. Case: $s + t > 0$

. If s, t have the same sign then $(\star)$ represents ellipse.
. If $s = 0$ or $t = 0$ then $(\star)$ represents two parallel lines.
. If $s = 1$ and $t = 1$ then $(\star)$ represents circle.
. If s, t have different sign then $(\star)$ represents hyperbola.

. Case: $s + t < 0$

. If s, t have the same sign then $(\star)$ represents ellipse.
. If $s = 0$ or $t = 0$ then $(\star)$ represents two parallel lines.
. If $s = 1$ and $t = 1$ then $(\star)$ represents circle.
. If s, t have different sign then $(\star)$ represents hyperbola.

. Case: $s + t = 0$ then $(\star)$ is equivalent to

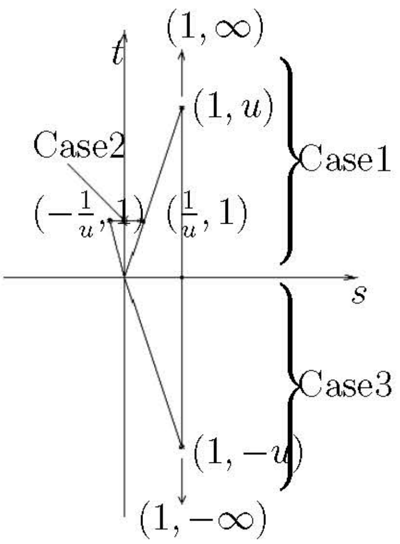
$$\begin{aligned} tx^2 + sy^2 &= 0 \\ -sx^2 + sy^2 &= 0 \\ y &= \pm x, \quad \textit{represents two crossing lines.} \end{aligned}$$

Firstly, we choose $s = 1$ and $t = 0$ then the curve become two parallel lines whose parallel on x-axis.
Case 1: If we fix $s = 1$ and move t up then the curve become ellipse which has major along x-axis. If we still move t up and up until $t = 1$ the curve become the circle and if we still move up and up the curve become ellipse which has major along y-axis.

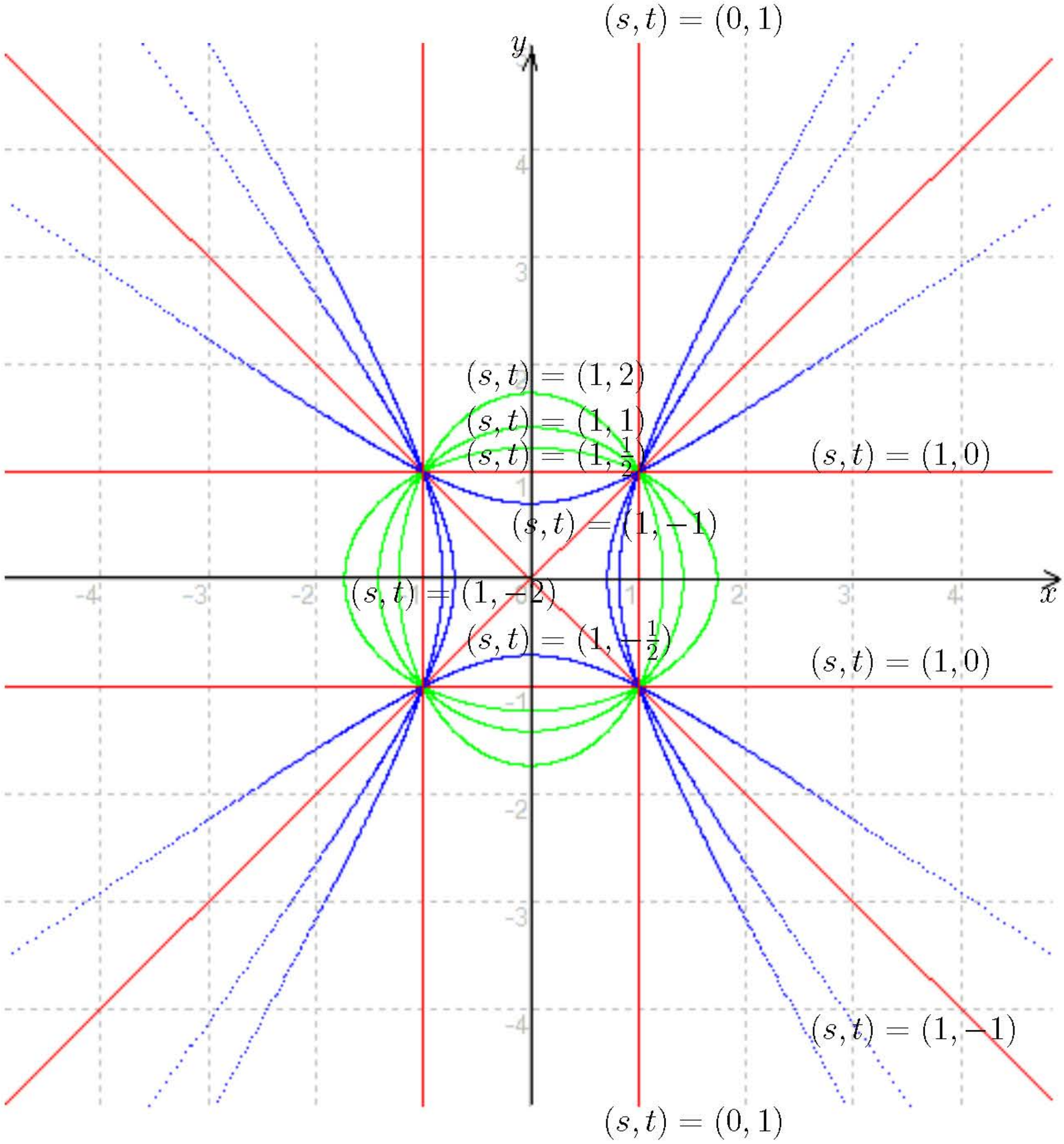
Case 2: When $s = 0$ and $t = 1$ then the curve become two parallel lines whose parallel on y-axis.

Case 3: Similarly, if we fix $s = 1$ and move t from 0 to down and down the curve become hyperbola which concave on y-axis untill $t = -1$ then the curve become two croosing lines. If we still move t down and down then we obtain hyperbola which concave on x-axis.
Given a pair (s, t), the equation $(\star)$ represents a figure. Different pairs (s, t) and (s', t') may represent the same figure.

Definition:
We say (s, t) is equivalent to (s', t') if they represent the same figure, i.e., the equation $(\star)$ coresponding to (s, t) is equivalent to the equation $(\star)$ coresponding to (s', t').
In other words, $(s, t) \sim (s', t') \Longleftrightarrow s : t = s' : t'$.
The above three cases can be regarded as single case, if we admit $t = \pm\infty$.
Because,
1) $(1, u) \sim (\frac{1}{u}, 1)$, likely $(1, -u) \sim (\frac{1}{-u}, 1)$,
2) when $u \to \infty$, then $(1, u) \to (1, \infty)$ and $(1, -u) \to (1, -\infty)$,
3) $(\frac{1}{u}, 1) \longrightarrow (0, 1)$ and $(\frac{1}{-u}, 1) \longrightarrow (0, 1)$,
we have $(1, \infty) \sim (1, -\infty)$.
Therefore, when t move from 0 to $+\infty$, the limit $(1, \infty)$ is equivalent to $(0, 1)$. On the other hand, when t move from 0 to $-\infty$, the limit $(1, -\infty)$ is equivalent to $(0, 1)$.
Thus, we can reqard as $-\infty = \infty$.
So t move from 0 to $+\infty$, which is equal to $-\infty$. Then t move from $-\infty$ to 0. Thus, t can move around all the cases continuously.
Similarly, we fix $t = 1$ and move s from 0 to $+\infty = -\infty$, and move up to 0, we get the same result.



Another Way: If we choose $s = \cos\theta$ and $t = \sin\theta$ and $0 \le \theta \le \pi$ then we get the same result.



Example6: Given five points $A(1,0), B(2,0), C(0,1), D(2,2), E(1,4)$. Find the equation of the quadratic curve passing through all of them.

Solution1:
Let the equation to be

$$\mathcal{C} : ax^2 + bxy + cy^2 + dx + ey + f = 0 \qquad (\star), \quad where \ \ (a,b,c) \neq (0,0).$$

By subtituting the coordinate to $(\star)$, we obtain
$\mathcal{C} \ni A(1,0)$ imply

$$a + d + f = 0 \qquad (1)$$

$\mathcal{C} \ni B(2,0)$ imply

$$4a + 2d + f = 0 \qquad (2)$$

$\mathcal{C} \ni C(0,1)$ imply

$$a + b + c + d + e + f = 0 \qquad (3)$$

$\mathcal{C} \ni D(2,2)$ imply

$$4a + 4b + 4c + 2d + 2e + 2f = 0 \qquad (4)$$

$\mathcal{C} \ni E(1,4)$ imply

$$a + 4b + 16c + d + 4e + f = 0 \qquad (5)$$

By solving these system equation, we obtain

$$\begin{aligned} a - \frac{1}{2}f &= 0 \\ b - \frac{2}{5}f &= 0 \\ c - \frac{1}{5}f &= 0 \\ d + \frac{3}{2}f &= 0 \\ e + \frac{12}{10}f &= 0 \end{aligned}$$

We choose $f = 20$ then

$$\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} = \begin{pmatrix} 10 \\ 8 \\ 4 \\ -30 \\ -24 \\ 20 \end{pmatrix}$$

Therefore the equation of $\mathcal{C} : 10x^2 + 8xy + 4y^2 - 30x - 24y + 20 = 0.$

Remark: This solution is complicated to calculate because we need to solve the system of equations. Sometimes the students can confuse the sign and it take long time to calculate it. To make how to solve this problem easier and shorter than, we have another solution.

Solution2:

By analogy of the case of pencil of line (see Excercise 4). The equation of $\mathcal{C}$

will be obtained by the linear combination of the equation of the curve pass through four points:

$$\mathcal{C}: \quad sf_{AB}(x,y)\cdot f_{CD} + tf_{AD}\cdot f_{BC} = 0 \quad for\ some\ (s,t)\in\mathbb{R}. \qquad (\star)$$

$$f_{AB}(x,y) = \begin{vmatrix} x - x_A & y - y_A \\ x_B - x_A & y_B - y_A \end{vmatrix} = \begin{vmatrix} x-1 & y \\ 1 & 0 \end{vmatrix} = -y$$

$$f_{CD}(x,y) = \begin{vmatrix} x - x_C & y - y_C \\ x_D - x_C & y_D - y_C \end{vmatrix} = \begin{vmatrix} x & y-1 \\ 2 & 1 \end{vmatrix} = x - 2(y-1)$$

$$f_{AD}(x,y) = \begin{vmatrix} x - x_A & y - y_A \\ x_D - x_A & y_D - y_A \end{vmatrix} = \begin{vmatrix} x-1 & y \\ 1 & 2 \end{vmatrix} = 2(x-1) - y$$

$$f_{BC}(x,y) = \begin{vmatrix} x - x_B & y - y_B \\ x_C - x_B & y_C - y_A \end{vmatrix} = \begin{vmatrix} x-2 & y \\ -2 & 1 \end{vmatrix} = (x-2) + 2y$$

then the equation of $(\star)$ become

$$s(-y)(x-2y+2) + t(2x-y-2)(x+2y-2) = 0$$

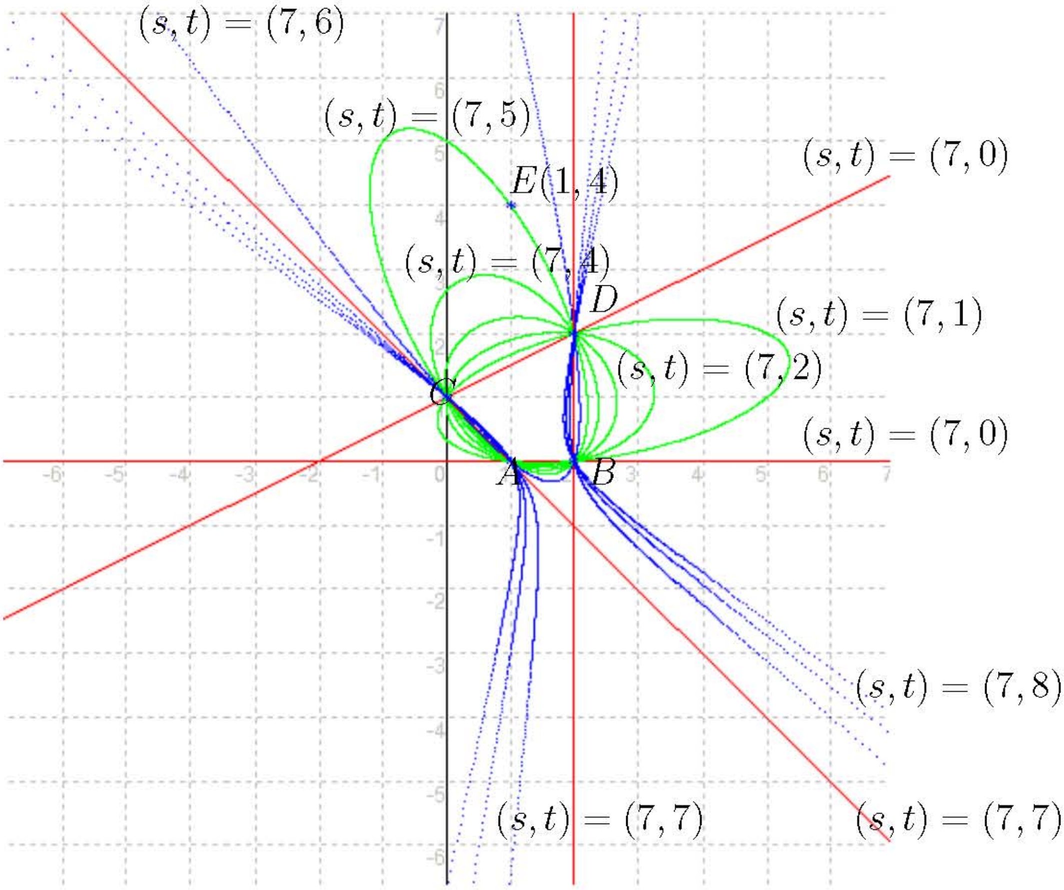
represent a quadratic curve pass through four points A, B, C, D.
The curve pass through $E(1,4)$ if and only if $E(1,4)$ satidfy $(\star)$

$$\begin{aligned} s(-4)(1 - 2\times 4 + 2) + t(2-4-2)(1+2\times 4 - 2) &= 0 \\ 20s - 28t &= 0 \\ 5s - 7t &= 0 \end{aligned}$$

We choose $s = 7$ and $t = 5$, then $(\star)$ is equivalent to

$$7(-y)(x-2y+2) + 5(2x-y-2)(x+2y-2) = 0$$

Therefore the equation of $\mathcal{C}:\ 10x^2 + 8xy + 4y^2 - 30x - 24y + 20 = 0.$



We choose $s = 7$ and $t = 0$, then we obtain two crossing lines whose passing through four points A, B, C, D.
Case 1: If we fix $s = 7$ and move t up then the curve become ellipse.
Case 2: If we still move t up untill $t = 5$ then the curve become ellipse and we see the ellipse pass through E at the point $E(7, 5)$ and if we still move up then we obtain other ellipse which pass through four points A, B, C, D.
Case 3: When we still move t up untill $t = 6$ then the curve become parabola which pass through four points A, B, C, D.
Case 4: when we move t up untill $t = 7$ then the curve become two crossing line which pass through four points A, B, C, D
Case 5: When we still move t up then the curve become hyperbola which pass through four points A, B, C, D.

Comparison: Thus, the solution 2 is easier and shorter than the solution 1 because it is easy to find the equation of the line pass through 2 points and then use the method to calculate the system of the linear combination by giving arbitrary constants (s, t), then we can find the equation of $\mathcal{C}$ easily. Therefore the solution 2 is an analogy of the solution 1.

4.4 Classification of Conics

In the Cartesian coordinate system, the graph of a quadratic equation in two variables is always a conic section though it may be degenerate, and all conic sections arise in this way. The equation will be of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \quad \textit{with } A, B, C \quad \textit{not all zero.} \cdots (\star)$$

As scaling all six constants yields the same locus of zeros, one can consider conics as points in the five-dimensional projective space.

The conic sections described by this equation can be classified with the discriminant

$$B^2 - 4AC.$$

If the conic is non-degenerate, then:

. If $B^2 - 4AC < 0$, the equation represents an ellipse,

(unless the conic is degenerate, for example $x^2 + y^2 + 10 = 0$, which has no real-valued solutions.);

. If $A = C$ and $B = 0$, the equation represents a circle;

. If $B^2 - 4AC = 0$, the equation represents a parabola;

. If $B^2 - 4AC > 0$, the equation represents a hyperbola;

. If we also have A + C = 0, the equation represents a rectangular hyperbola.

Note that A and B are just polynomial coefficients, not the lengths of semi-major/minor axis as defined in the following sections.

a) In matrix notation the equation $(\star)$ becomes:

$$\begin{bmatrix} x & y \end{bmatrix} . \begin{bmatrix} A & \frac{B}{2} \\ \frac{B}{2} & C \end{bmatrix} . \begin{bmatrix} x \\ y \end{bmatrix} + Dx + Ey + F = 0$$

or

$$\begin{bmatrix} x & y & 1 \end{bmatrix} . \begin{bmatrix} A & \frac{B}{2} & \frac{D}{2} \\ \frac{B}{2} & C & \frac{E}{2} \\ \frac{D}{2} & \frac{C}{2} & F \end{bmatrix} . \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$

and

$$B^2 - 4AC = -4 \begin{bmatrix} A & \frac{B}{2} \\ \frac{B}{2} & C \end{bmatrix}.$$

This is the usual way of representing a quadratic form by a symmetric matrix, while the factor of -4 is a constant due to the use of $B^2 - 4AC$ rather than $AC - \frac{B^2}{4}$ in the quadratic formula.

b) As slice of quadratic form, The equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

can be rearranged by taking the affine linear part to the other side, yielding

$$Ax^2 + Bxy + Cy^2 = -(Dx + Ey + F).$$

In this form, a conic section is realized exactly as the intersection of the graph of the quadratic form $z = Ax^2+Bxy+Cy^2$ and the plane $z = -(Dx+Ey+F)$. Parabola and hyperbola can be realized by a horizontal plane ($D = E = 0$), while ellipses require that the plane be slanted. Degenerate conics correspond to degenerate intersections, such as taking slices such as $z = -1$ of a positive definite form.

c) Through change of coordinates these equations can be put in standard forms:

. Circle: $x^2 + y^2 = a^2$
. Ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
. Parabola: $y^2 = 4ax, \quad x^2 = 4ay$
. Hyperbola: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad \frac{x^2}{b^2} - \frac{y^2}{a^2} = -1$
. Rectangular Hyperbola: $xy = c^2$

Such forms will be symmetrical about the x-axis and for the circle, ellipse and hyperbola symmetrical about the y-axis. The rectangular hyperbola however is only symmetrical about the lines y = x and y = - x.
Therefore its inverse function is exactly the same as its original function.
These standard forms can be written as parametric equations,

. Circle: $(a\cos\theta, a\sin\theta)$,
. Ellipse: $(a\cos\theta, b\sin\theta)$,

. Parabola: $(at^2, 2at)$,
. Hyperbola: $(a\sec\theta, b\tan\theta)$ or $(\pm a\cos hu, b\sin hu)$.
. Rectangular hyperbola: $(ct, \frac{c}{t})$

d) For some practical applications, it is important to re-arrange the standard form so that the focal-point can be placed at the origin. The mathematical formulation for a general conic section is then given in the polar form by

$$r = \frac{\ell}{1 - e\cos\theta}$$

and in the Cartesian form by

$$\sqrt{x^2 + y^2} = (\ell + ex)$$

$$\Rightarrow \left(\frac{x - \dfrac{\ell e}{1-e^2}}{\dfrac{\ell}{1-e^2}} \right)^2 + \frac{(1-e^2)y^2}{\ell^2} = 1$$

From the above equation, the linear eccentricity (c) is given by $c = \left(\frac{\ell e}{1-e^2} \right)$.
From the general equations given above, different conic sections can be represented as shown below:

. Circle: $x^2 + y^2 = r^2$
. Ellipse: $\frac{\left(x - \sqrt{a^2 - b^2}\right)^2}{a^2} + \frac{y^2}{b^2} = 1$
. Parabola: $y^2 = 4a(x + a)$
. Hyperbola: $\frac{\left(x + \sqrt{a^2 + b^2}\right)^2}{a^2} - \frac{y^2}{b^2} = 1$

e) In homogeneous coordinates a conic section can be represented as:

$$A_1x^2 + A_2y^2 + A_3z^2 + 2B_1xy + 2B_2xz + 2B_3yz = 0.$$

Or in matrix notation $\begin{bmatrix} x & y & z \end{bmatrix} . \begin{bmatrix} A_1 & B_1 & B_2 \\ B_1 & A_2 & B_3 \\ B_2 & B_3 & A_3 \end{bmatrix} . \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

The matrix $M = \begin{bmatrix} A_1 & B_1 & B_2 \\ B_1 & A_2 & B_3 \\ B_2 & B_3 & A_3 \end{bmatrix}$ is called the matrix of the conic section.

$\Delta = det(M) \ = \ det\left(\begin{bmatrix} A_1 & B_1 & B_2 \\ B_1 & A_2 & B_3 \\ B_2 & B_3 & A_3 \end{bmatrix}\right)$ is called the determinant of the conic section.

If $\Delta = 0$ then the conic section is said to be degenerate, this means that the conic section is in fact a union of two straight lines. A conic section that intersects itself is always degenerate, however not all degenerate conic sections intersect themselves, if they do not they are straight lines.

For example, the conic section $\begin{bmatrix} x & y & z \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} . \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$ reduces to the union of two lines:

$$\{x^2 - y^2 = 0\} = \{(x + y)(x - y) = 0\} = \{x + y = 0\} \cup \{x - y = 0\}.$$

Similarly, a conic section sometimes reduces to a (single) line:

$\{x^2+2xy+y^2 = 0\} = \{(x+y)^2 = 0\} = \{x+y = 0\}\cup\{x+y = 0\} = \{x+y = 0\}.$

$\delta = det\left(\begin{bmatrix} A_1 & B_1 \\ B_1 & A_2 \end{bmatrix}\right)$ is called the discriminant of the conic section.

. If $\delta > 0$, it is an ellipse.
. If $\delta = 0$ then the conic section is a parabola.
. If $\delta < 0$, it is an hyperbola.
. If $\delta > 0$ and $A_1 = A_2$ and $B_1 = 0$ then the conic section is a circle.
. If $\delta < 0$ and $A_1 = -A_2$ then the conic section is an rectangular hyperbola.

It can be proven that in the complex projective plane of two conic sections have four points in common (if one accounts for multiplicity), so there are never more than 4 intersection points and there is always 1 intersection point (possibilities: 4 distinct intersection points, 2 singular intersection points and 1 double intersection points, 2 double intersection points, 1 singular intersection point and 1 with multiplicity 3, 1 intersection point with multiplicity 4). If there exists at least one intersection point with multiplicity > 1, then the two conic sections are said to be tangent. If there is only one intersection point, which has multiplicity 4, the two conic sections are said to be osculating.

Furthermore each straight line intersects each conic section twice. If the intersection point is double, the line is said to be tangent and it is called the tangent line. Because every straight line intersects a conic section twice, each conic section has two points at infinity (the intersection points with the line at infinity). If these points are real, the conic section must be a hyperbola, if they are imaginary conjugated, the conic section must be an ellipse, if the conic section has one double point at infinity it is a parabola. If the points at infinity are (1,i,0) and (1,−i,0), the conic section is a circle. If a conic section has one real and one imaginary point at infinity or it has two imaginary points that are not conjugated it is neither a parabola nor an ellipse nor a hyperbola.

f) In polar coordinates, a conic section with one focus at the origin and, if any, the other on the x-axis, is given by the equation

$$r = \frac{\ell}{1 \pm e\cos\theta}$$

where e is the eccentricity and ℓ is the semi-latus rectum. As above, for $e = 0$, we have a circle, for $0 < e < 1$ we obtain an ellipse, for $e = 1$ a parabola, and for $e > 1$ a hyperbola.

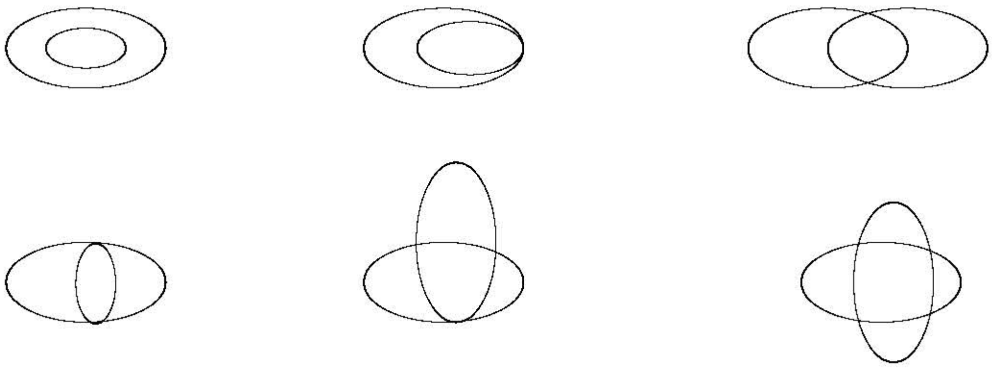
4.5 Intersection of Two Conics

The solutions to a two second degree equations system in two variables may be seen as the coordinates of the intersections of two generic conic sections. In particular two conics may possess none, two or four possibly coincident intersection points. The best method of locating these solutions exploits the homogeneous matrix representation of conic sections, i.e. 3×3 symmetric matrix which depends on six parameters.
The procedure to locate the intersection points follows these steps:

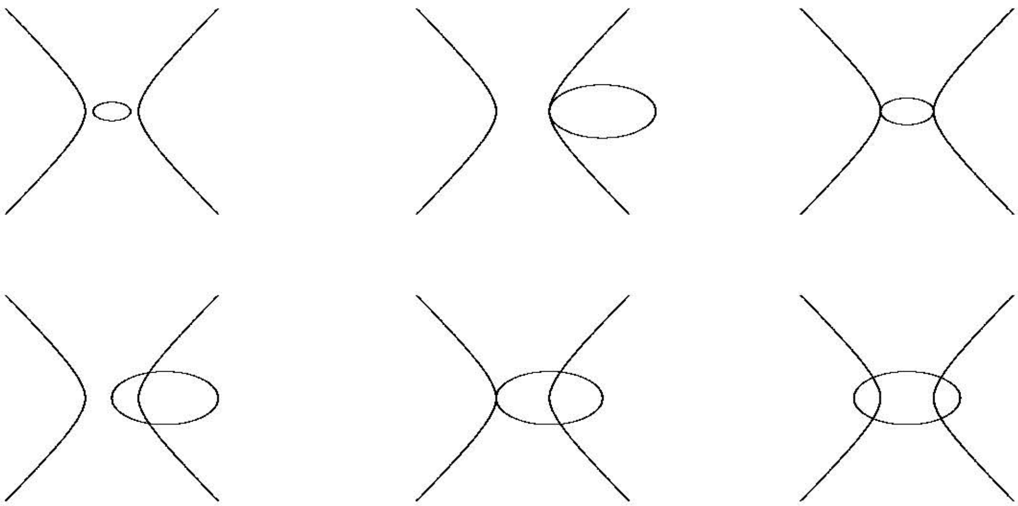
- Given the two conics C_1 and C_2 consider the pencil of conics given by their linear combination $\lambda C_1 + \mu C_2$
- Identify the homogeneous parameters (λ, μ) which corresponds to the degenerate conic of the pencil. This can be done by imposing that $det(\lambda C_1 + \mu C_2) = 0$, which turns out to be the solution to a third degree equation.
- Given the degenerate conic C_0, identify the two, possibly coincident, lines constituting it.
- Intersects each identified line with one of the two original conic; this step can be done efficiently using the dual conic representation of C_0
- The points of intersection will represent the solution to the initial equation system.

Now, we consider about all the case of conic as below:

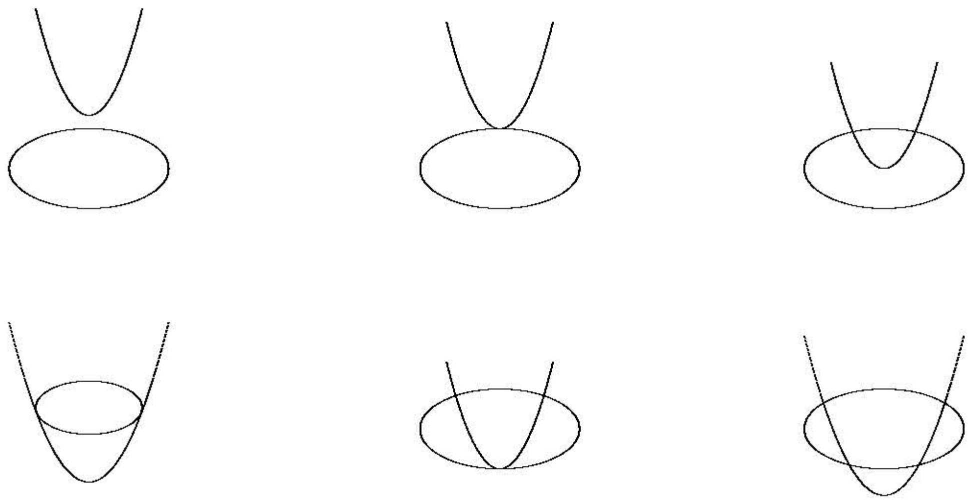
4.5.1 Case- Ellipse and Ellipse



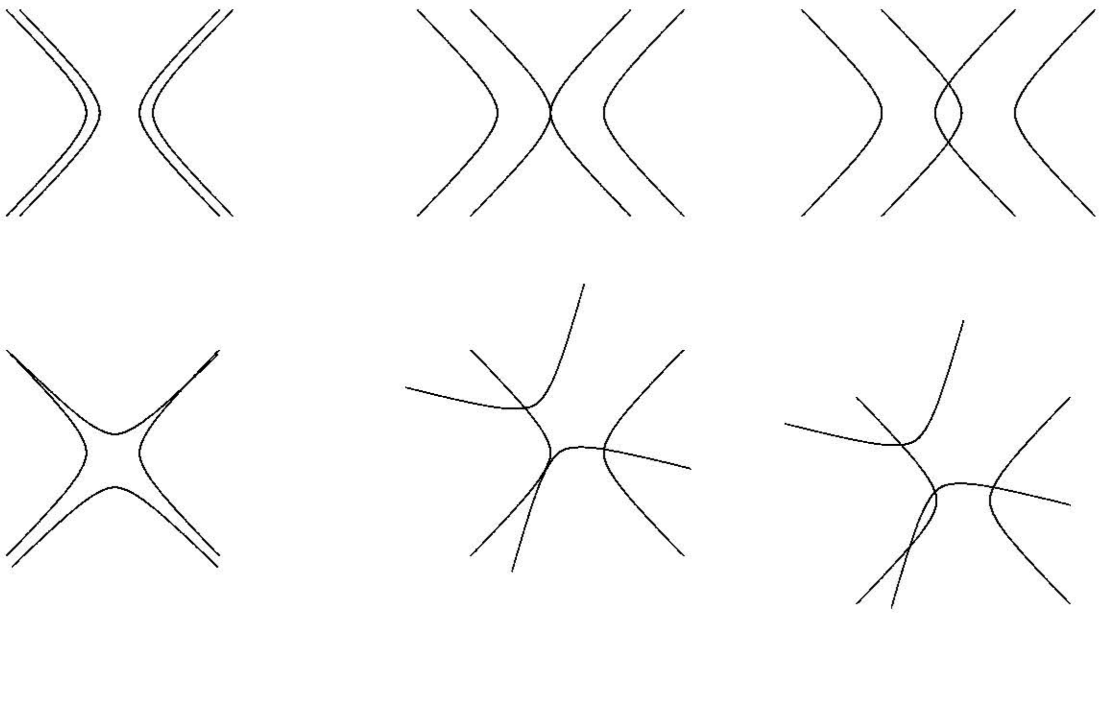
4.5.2 Case- Ellipse and Hyperbola



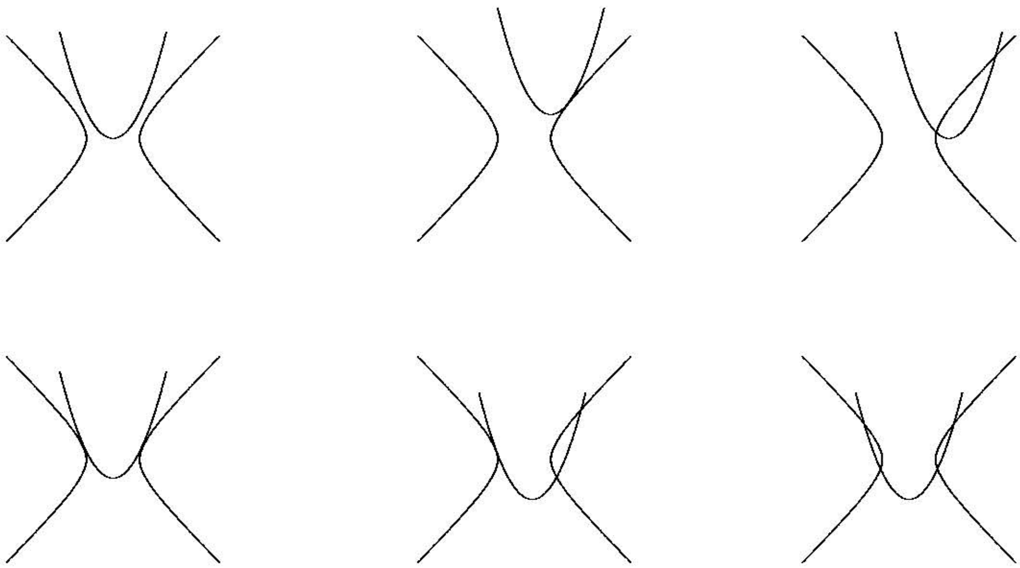
4.5.3 Case- Ellipse and Parabola



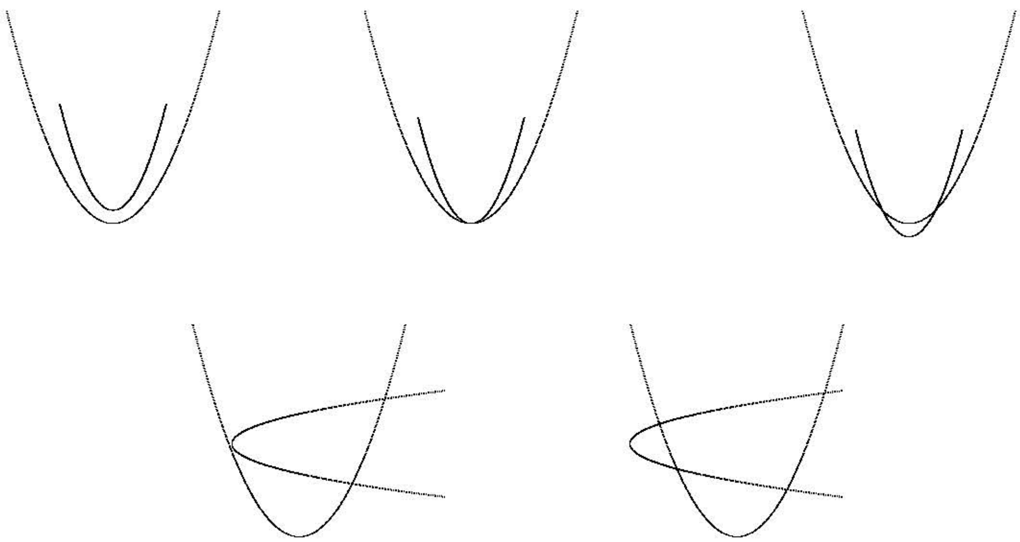
4.5.4 Case- Hyperbola and Hyperbola



4.5.5 Case- Hyperbola and Parabola



4.5.6 Case- Parabola and Parabola



Remark:

The common point of the intersection of two conics are :

. No common point(two conics do not intersect with each other).
. One common point (two conics are tangent to each other).
. Two common points

.Two conics cross each other at two points.
.Two conics tangent to each other at two points.

. Three common points (two conics are intersect at two points and tagent each other at one point).
. Four common points (two conics are intersect at four points)

Chapter 5

Making and Solving Problems

5.1 Comparison of Analytic Method and Synthetic Method:

In these section, we compare two methods in geometry: analytic method and synthetic method.
In analytic method, we should always pay attention whether coordinate axes are given or not. If coordinate axes are not given in the problem, we must choose the coordinate axes by ourselves. Note that they can be choosen as we like.
Problem1: Given a circle with radius a and center at point F. Take another fixed point F' inside this circle and take Q as a point moving along the circumference. Take P as the intersection point of QF and the perpendicular bisector of QF'. Then the locus of P is an ellipse with F and F' as its foci. Prove this statement.
Solution:(Analytic method)
Let $F = (0, 0)$ and $F' = (b, 0)$, $|b| < a$.
Then the equation of the circle is

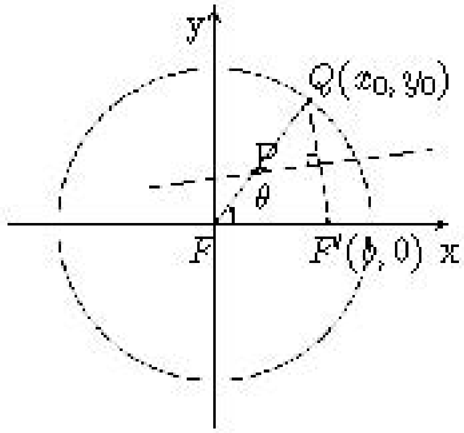
$$x^2 + y^2 = a^2$$

Let $\theta =$ be the angle $\angle F'FQ$, then
$Q = (a \cdot \cos\theta, a \cdot \sin\theta)$
Let ℓ is perpendicular bisector of QF' then equation of QF is

$$\begin{aligned} \overrightarrow{QF} // \overrightarrow{FX} &\iff x_0 y - x y_0 = 0 \\ &\iff (a.\cos\theta)y - (a.\sin\theta)x = 0 \end{aligned}$$

then $QF : (\cos\theta)y - (\sin\theta)x = 0$
then the equation of ℓ is

$$2(a\cos\theta - b)x + 2(a\sin\theta)y = (a\cos\theta)^2$$



Therefore $\ell : 2(a\cos\theta - b)x + 2(a\sin\theta)y = a^2 - b^2$
let P(x,y) is the intersection point.
then

$$\begin{cases} (\cos\theta)y - (\sin\theta)x = 0 \\ (a\cos\theta - b)x + (a\sin\theta)y = \dfrac{a^2-b^2}{2} \end{cases}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{-a\sin^2\theta - a\cos^2\theta + b\cos\theta} \begin{pmatrix} a\sin\theta & -(a\cos\theta - b) \\ -\cos\theta & -\sin\theta \end{pmatrix} \begin{pmatrix} 0 \\ \dfrac{a^2-b^2}{2} \end{pmatrix}$$

$$= \frac{1}{-a + b\cos\theta} \begin{pmatrix} -\dfrac{a^2-b^2}{2}\cos\theta \\ -\dfrac{a^2-b^2}{2}\sin\theta \end{pmatrix}$$

then

$$\begin{cases} x = \dfrac{1}{-a + b\cos\theta}\left(-\dfrac{a^2-b^2}{2}\right)\cos\theta \Longrightarrow x = \dfrac{a^2-b^2}{2(a - b\cos\theta)\cos\theta} \\ y = \dfrac{1}{-a + b\cos\theta}\left(-\dfrac{a^2-b^2}{2}\right)\sin\theta \Longrightarrow y = \dfrac{a^2-b^2}{2(a - b\cos\theta)}\sin\theta \end{cases}$$

we must eliminate θ

$$x^2 + y^2 = \left(\frac{a^2-b^2}{2(a - b\cos\theta)}\right)^2 \qquad (\star)$$

$$x = \frac{a^2-b^2}{2(a - b\cos\theta)\cos\theta}$$

$$2ax - 2bx\cos\theta = (a^2 - b^2)\cos\theta$$

$$2ax = (a^2 - b^2 + 2bx)\cos\theta$$

$$\begin{aligned}
\cos\theta &= \frac{2ax}{a^2-b^2+2bx} \\
a-b\cos\theta &= a-\frac{2abx}{a^2-b^2+2bx} \\
&= \frac{a^3-ab^2+2abx-2abx}{a^2-b^2+2bx} \\
&= \frac{(a^2-b^2)a}{a^2-b^2+2bx} \\
\frac{a^2-b^2}{2(a-b\cos\theta)} &= \frac{a^2-b^2}{2}\times\frac{1}{(a-b\cos\theta)} \\
&= \frac{(a^2-b^2)}{2}\times\frac{a^2-b^2+2bx}{(a^2-b^2)a} \\
&= \frac{a^2-b^2+2bx}{2a}
\end{aligned}$$

Therefore

$$\begin{aligned}
(\star) \iff x^2+y^2 &= \frac{(a^2-b^2+2bx)^2}{4a^2} \\
4a^2x^2+4a^2y^2 &= (a^2-b^2+2bx)^2 \\
4a^2x^2+4a^2y^2 &= (a^2-b^2)^2+4(a^2-b^2)bx+4b^2x^2 \\
4a^2x^2+4a^2y^2 &= (a^2-b^2)^2+4(a^2-b^2)bx+4b^2x^2 \\
4(a^2-b^2)x^2-4(a^2-b^2)bx+4a^2y^2 &= (a^2-b^2) \\
x^2-bx+\frac{a^2}{a^2-b^2}y^2 &= \frac{a^2-b^2}{4} \\
(x-\frac{b}{2})^2-\frac{b^2}{4}+\frac{a^2}{a^2-b^2}y^2 &= \frac{a^2-b^2}{4} \\
(x-\frac{b}{2})^2+(\frac{a^2}{a^2-b^2}y^2 &= \frac{a^2}{4} \\
\frac{4(x-\frac{b}{2})^2}{a^2}+\frac{4a^2}{(a^2-b^2)a^2}y^2 &= 1
\end{aligned}$$

Therefore the locus of P is a part of ellipse.

$$\frac{(x-\frac{b}{2})^2}{(\frac{a}{2})^2}+\frac{y^2}{\frac{(\sqrt{a^2-b^2})^2}{2}} = 1.$$

Remark:

Similarly problem 1, if F' is out of the circle, then $a^2-b^2<0$.

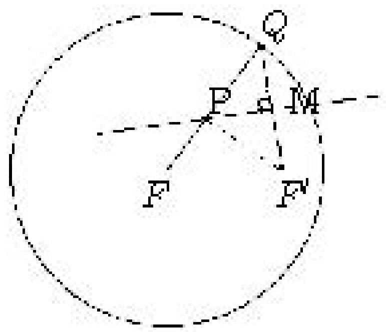
Therefore, since $a^2 - b^2 = -(b^2 - a^2)$, the locus of P is a hyperbola.

$$\frac{(x-\frac{b}{2})^2}{(\frac{a}{2})^2} - \frac{y^2}{\frac{(\sqrt{b^2-a^2})^2}{2}} = 1.$$

(Synthetic method)

Let M be the midpoint of QF'.

Since P is the intersection of QF and perpendicular bisector of QF', we have



$$\angle PMQ = \angle PMF'$$
$$PM = PM$$
$$MQ = MF'$$

Therefore $\triangle PMQ \equiv \triangle PMF'$

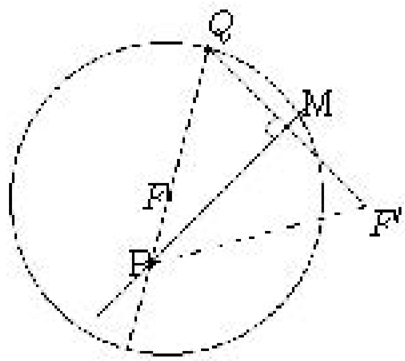
Thus, $PQ = PF'$

$FP + PF' = FP + PQ = a$, which is

Therefore, the locus of the point P is

Remark:

Similarly, if F' is out of the circle, then



$PF + FQ = PF'$

$PF - PF' = a$, which is constant.

Therefore, the locus of the point P is and hyperbola with foci F, F'.

Comparison:

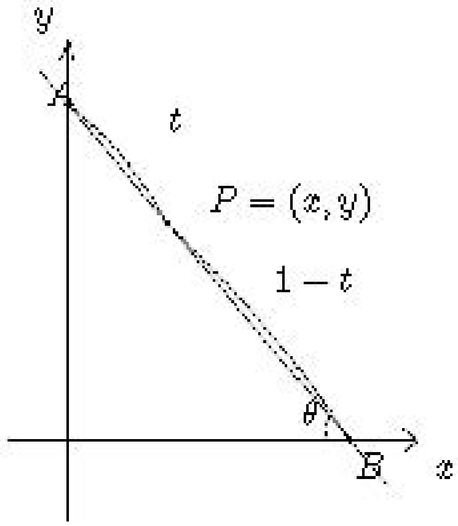
Among two methods above, we can compare that, the solution solved by synthetic method is more easier than analytic method because we only pompare two triangles, and then we can find the locus of the point P. But in the analytic method we need to calculate by using coordinate of the points. So it is very complicated to solve the above problem by using analytic method. But by analytic method, we can find the the equation of quadratic curve, from which we can know wether the curve is ellipe or hyperbola easily.

Problem2:

Given two orthogonal lines ℓ and m, and a segment AB with length 1 such that A is on ℓ and B is on m. Let P be a point fixed on AB. If we move A and B, then find the locus of point P.

Solution:

Choose x-axis to be m and y-axis to be ℓ.
Since, $AB = 1$, then we obtain the coordinate of points A and B as follow:
$A = (0, \sin\theta)$, $B = (\cos\theta, 0)$
The point $P = (x, y)$ is a point fixed on AB, so that we can suppose that $AP : PB = t : (1-t)$, where $0 < t < 1$, and t is constant. Then we get



$$\overrightarrow{OP} = (1-t)\overrightarrow{OA} + t\overrightarrow{OB}$$
$$= (1-t)\begin{bmatrix} 0 \\ \sin\theta \end{bmatrix} + t\begin{bmatrix} \cos\theta \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} t\cos\theta \\ (1-t)\sin\theta \end{bmatrix}.$$

On the other hand $\overrightarrow{OP} = \begin{bmatrix} x \\ y \end{bmatrix}$.
Thus $\cos\theta = \frac{x}{t}$, $\quad \sin\theta = \frac{y}{1-t}$.
Therefore, the locus of P is an ellipse

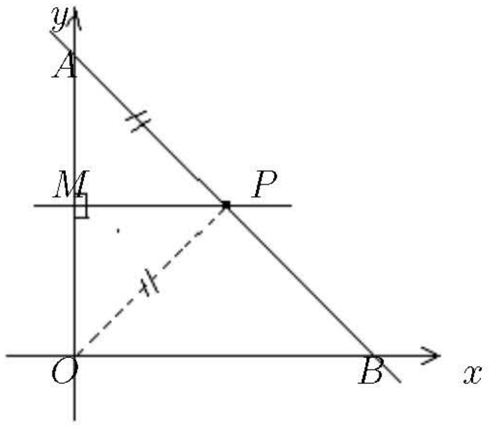
$$\frac{x^2}{t^2} + \frac{y^2}{(1-t)^2} = 1$$

Remark 1:

As a special case of this problem, when P is the midpoint of AB, the locus of P is a circle. This can be shown by synthetic method as follows:
Draw a line passing through P which is parallel to x-axis, whence perpendicular to OA.
Let M be the intersection of the line passing through P and OA. We have

$$\angle OMP = \angle AMP$$
$$MP = MP$$
$$OM = AM$$

because the point P is the midpoint of AB and M is the intersection of the line passing through P and parallel to x-axis. Hence $OM = AM$.(Converse of two midpoint theorem)
Then $\triangle OMP \equiv \triangle AMP$.
Thus, $OP = AP = \dfrac{AB}{2} = \dfrac{1}{2}$.
Therefore the locus of the point P is the circle with center O and radius $\dfrac{1}{2}$.



Remark 2:
Similarly, in the case P is an external dividing point of AB, the locus of P is also an ellipse. We can prove it similarly to the proof of this problem. We just suppose that $t > 1$ is enough to prove this problem for external division point. So, in this problem, external case is analogous to internal case.

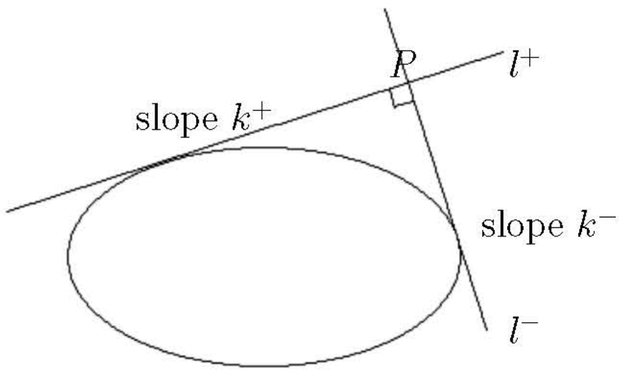
5.2 Analogy:

In section 1 we have an example of analogy (see Remark 2).

we will give the another example of analogy.

Problem3
Given an ellipse $\mathcal{E} : \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Fin wo
tangent lines to ellipse are perpendicul

Solution:
We have $\mathcal{E} : \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \qquad (1)$
Let $P(x_1, y_1)$
Let ℓ be a line passing through P with slope k.
Then the equation of ℓ is
$\ell : y - y_1 = k(x - x_1) \qquad (2)$



To find the condition on k such that ℓ tangents to $\mathcal{E}$, we substitute (2) to (1), then

$$\frac{x^2}{a^2} + \frac{(k(x - x_1) + y_1)^2}{b^2} = 1$$
$$b^2x^2 + a^2\{k^2(x - x_1)^2 + 2k(x - x_1)y_1 + y_1^2\} = a^2b^2$$
$$b^2x^2 + a^2k^2x^2 - 2a^2k^2x_1x + a^2k^2x_1^2 + 2a^2kxy_1 - 2a^2kx_1y_1 + a^2y_1^2 = a^2b^2$$
$$(b^2 + a^2k^2)x^2 - 2a^2k(kx_1 - y_1)x + a^2k^2x_1^2 - 2a^2kx_1y_1 + a^2y_1^2 - a^2b^2 = 0$$

Let D be the discriminant, then

$$\begin{aligned}\frac{D}{4} &= (a^2k(kx_1 - y_1))^2 - (b^2 + a^2k^2)(a^2k^2x_1^2 - 2a^2kx_1y_1 + a^2y_1^2 - a^2b^2) \\ &= a^2b^2((a^2 - x_1^2)k^2 + 2x_1y_1k - y_1^2 + b^2)\end{aligned}$$

Since ℓ is tangent to $\mathcal{E}$, we have $D = 0$. Then, since $ab \neq 0$, we obtain

$$(a^2 - x_1^2)k^2 + 2x_1y_1k - y_1^2 + b^2 = 0 \quad (\star)$$

Reqarding $(\star)$ as an equation with respect to k, we obtain two values of k:

$$\begin{aligned}k^{\pm} &= \frac{-x_1y_1 \pm \sqrt{x_1^2y_1^2 + (a^2 - x_1^2)(y_1^2 - b^2)}}{(a^2 - x_1^2)} \\ &= \frac{-x_1y_1 \pm \sqrt{a^2y_1^2 + b^2x_1^2 - a^2b^2}}{(a^2 - x_1^2)}\end{aligned}$$

Let $\ell^{\pm}$ be the line ℓ with slope $k^{\pm}$.
Two lines are perpendicular with each other if and only if the product of their slopes is equal to -1. (i.e,. $\ell^+ \perp \ell^- \iff k^+ \times k^- = -1$).
then

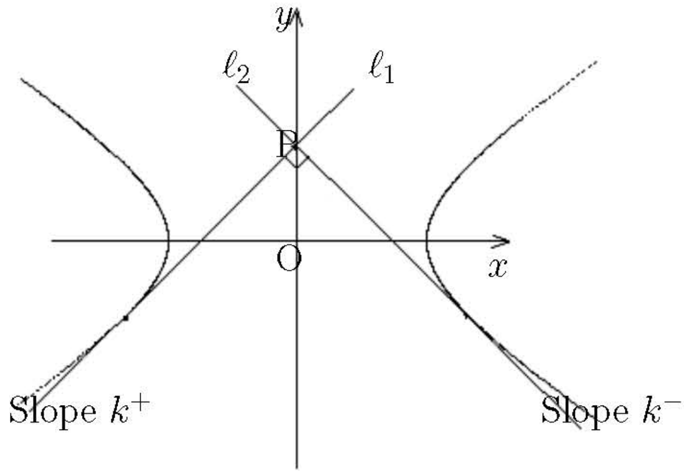
$$\frac{-x_1y_1 + \sqrt{a^2y_1^2 + b^2x_1^2 - a^2b^2}}{(a^2 - x_1^2)} \times \frac{-x_1y_1 - \sqrt{a^2y_1^2 + b^2x_1^2 - a^2b^2}}{(a^2 - x_1^2)} = -1$$
$$\frac{y_1^2 - b^2}{x_1^2 - a^2} = -1$$
$$y_1^2 - b^2 = -x_1^2 + a^2$$
$$y_1^2 + x_1^2 = a^2 + b^2$$

Therefore the locus of P is a circle with center O and radius$\sqrt{a^2 + b^2}$.

As the analogy to the problem5, we consider on hyperbola:

Problem4: Given an hyperbola $\mathcal{H} : \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$. Find the locus of point P from which two tangent lines to hyperbolar are perpendicular to each other.

Problem6 can be solved in the similar way to the solution of the problem3. We just only change the sign to the equation of hyperbola, then we get the locus of the point P is a circle with center O and radius $\sqrt{a^2 - b^2}$. In order to show this, we look back the previous solution.



Solution(revised):

We have $\mathcal{E} : \dfrac{x^2}{a^2} + c\dfrac{y^2}{b^2} = 1 \qquad (1)$

Let $P(x_1, y_1)$

Let ℓ be a line passing through P with slope k.

Then the equation of ℓ is $\ell : y - y_1 = k(x - x_1) \qquad (2)$

To find the condition on k such that ℓ tangents to $\mathcal{E}$, we substitute (2) to (1), then

$$\frac{x^2}{a^2} + c\frac{(k(x-x_1)+y_1)^2}{b^2} = 1$$
$$b^2x^2 + ca^2\{k^2(x-x_1)^2 + 2k(x-x_1)y_1 + y_1^2\} = a^2b^2$$
$$b^2x^2 + ca^2k^2x^2 - 2ca^2k^2x_1x + ca^2k^2x_1^2 + 2ca^2ky_1x - 2ca^2kx_1y_1 + ca^2y_1^2 = a^2b^2$$
$$(b^2 + ca^2k^2)x^2 - 2ca^2k(kx_1 - y_1)x + ca^2k^2x_1^2 - 2ca^2kx_1y_1 + ca^2y_1^2 - a^2b^2 = 0$$

Let D be the discriminant, then

$$\begin{aligned}\frac{D}{4} &= (ca^2k(kx_1 - y_1))^2 - (b^2 + ca^2k^2)(ca^2k^2x_1^2 - 2ca^2kx_1y_1 + ca^2y_1^2 - a^2b^2) \\ &= ca^2b^2((a^2 - x_1^2)k^2 + 2x_1y_1k - y_1^2 + \frac{b^2}{c})\end{aligned}$$

Since ℓ is tangent to $\mathcal{E}$, we have $D = 0$. Then, since $abc \neq 0$, we obtain

$$(a^2 - x_1^2)k^2 + 2x_1y_1k - y_1^2 + \frac{b^2}{c} = 0 \quad (\star)$$

Reqarding $(\star)$ as an equation with respect to k, we obtain two values of k:

$$k^{\pm} = \frac{-x_1y_1 \pm \sqrt{x_1^2y_1^2 + (a^2 - x_1^2)(y_1^2 - \frac{b^2}{c})}}{(a^2 - x_1^2)} = \frac{-x_1y_1 \pm \sqrt{\dfrac{ca^2y_1^2 + b^2x_1^2 - a^2b^2}{c}}}{(a^2 - x_1^2)}$$

Let $\ell^{\pm}$ be the line ℓ with slope $k^{\pm}$.
Two lines are perpendicular with each other if and only if the product of their slopes is equal to -1. (i.e,. $\ell^{+} \perp \ell^{-} \iff k^{+} \times k^{-} = -1$).
then

$$\frac{-x_1y_1 + \sqrt{\dfrac{ca^2y_1^2 + b^2x_1^2 - a^2b^2}{c}}}{(a^2 - x_1^2)} \times \frac{-x_1y_1 - \sqrt{\dfrac{ca^2y_1^2 + b^2x_1^2 - a^2b^2}{c}}}{(a^2 - x_1^2)} = -1$$

$$\frac{c(x_1^2 - a^2)y_1^2 - (x_1^2 - a^2)b^2}{c(a^2 - x_1^2)^2} = -1$$

$$cy_1^2 + cx_1^2 = ca^2 + b^2$$

When $c = 1$, then the locus of P is a circle with center O and radius $\sqrt{a^2 + b^2}$.
When $c = -1$, then the locus of P is a circle with center O and radius $\sqrt{a^2 - b^2}$.

Problem5:

Let $\mathcal{E}$ be the ellipse $\frac{x^2}{4} + y^2 = 1 \cdots (1)$. Prove that, if the stright line $y = \frac{1}{2}x + k \cdots (2)$ and $\mathcal{E}$ have two intersections P and Q, the midpoint R of line seqment PQ lies on the line $y = -\frac{1}{2}x$ in the part corresponding to $-\sqrt{2} < x < \sqrt{2}$.

Solution:

To find the coordinates of P and Q,
substitute (2) to (1), then we obtain

$$\frac{x^2}{4} + (\frac{1}{2}x + k)^2 = 1$$

$$\frac{x^2}{4} + \frac{1}{4}x^2 + kx + k^2 - 1 = 0$$

$$\frac{1}{2}x^2 + kx + k^2 - 1 = 0$$

$$x^2 + 2kx + 2(k^2 - 1) = 0$$

Let D be the discriminant of the equation above, then
(1) and (2) have two common points if and only if $D > 0$, then
$\dfrac{D}{4} = k^2 - 2(k^2 - 1) > 0$
$-k^2 + 2 > 0 \Longleftrightarrow k^2 < 2$
$-\sqrt{2} < k < \sqrt{2}$
Put $P(x_1, y_1)$; $Q(x_2, y_2)$, $R(x, y)$, then

$$\begin{aligned}
x_1 + x_2 &= -2k \\
y_1 &= \frac{1}{2}x_1 + k \\
y_2 &= \frac{1}{2}x_2 + k \\
y_1 + y_2 &= \frac{x_1 + x_2}{2} + 2k \\
&= \frac{-2k}{2} + 2k \\
&= -k + 2k \\
&= k
\end{aligned}$$

The locus of point R is the midpoint of PQ, then
$R = \dfrac{P+Q}{2} = \left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right) = \left(-k, \dfrac{k}{2}\right)$
Thus, $x = -k$; $y = \dfrac{k}{2}$, so R is on the line $y = -\dfrac{1}{2}x$.
From $x = -k$ and $-\sqrt{2} < k < \sqrt{2}$, we obtain $-\sqrt{2} < x < \sqrt{2}$.
Therefore, R lies on the line $y = -\dfrac{1}{2}x$ in the part corresponding to $-\sqrt{2} < x < \sqrt{2}$.

Remark: What we have prove above is equivalent to say that *the locus of R is contained in the line $y = -\dfrac{1}{2}x$ in the part corresponding to $-\sqrt{2} < x < \sqrt{2}$.*
If we are asked to prove that *the locus of R is exactly the line $y = -\dfrac{1}{2}x$ in the part corresponding to $-\sqrt{2} < x < \sqrt{2}$*, the above argument is not complete. We must show the converse; i.e.,
given arbitrary point S on the line $y = -\dfrac{1}{2}x$ in the part corresponding to $-\sqrt{2} < x < \sqrt{2}$, we must show $R = S$ for some k.
The converse is proved as follow:

From the assumption, we get $S = (t, -\frac{1}{2})$ for some $-\sqrt{2} < t < \sqrt{2}$.
Choose $k = -t$, then, as we have shown above,

$$R = (-k, \frac{k}{2}) = (t, -\frac{t}{2}) = S$$

Problem6:
Let $\mathcal{H}$ be the hyperbola $x^2 - y^2 = 1 \cdots (1)$. Prove that, if the stright line $y = \frac{1}{2}x + k \cdots (2)$ and $\mathcal{H}$ have two intersections P and Q, the midpoint R of line seqment PQ lies on the line $y = 2x$.
Solution:
To find the coordinates of P and Q,
substitute (2) to (1), then we obtain

$$x^2 - (\frac{1}{2}x + k)^2 = 1$$
$$x^2 - \frac{1}{4}x^2 - kx - k^2 = 1$$
$$\frac{3}{4}x^2 - kx - (k^2 + 1) = 0$$

Let D be the descriminant of the equation above, then
(1) and (2) have two common points if and only if $D > 0$, then

$$\frac{D}{4} = k^2 + 3k^2 + 1 > 0$$
$$4k^2 > -1$$
$$k^2 > -\frac{1}{4} \quad \textit{it is true for all values of } k.$$

In order to find the coordinate of R,
put $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x, y)$

$$x_1 + x_2 = \frac{4}{3}k$$
$$y_1 = \frac{1}{2}x_1 + k$$
$$y_2 = \frac{1}{2}x_2 + k$$
$$y_1 + y_2 = \frac{x_1 + x_2}{2} + 2k = \frac{8}{3}k$$

The locus of R is the midpoint of PQ, then
$R = \frac{P+Q}{2} = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \left(\frac{2}{3}k, \frac{4}{3}k\right)$
Thus, the coordinate of $R(\frac{2}{3}k, \frac{4}{3}k)$
The locus of R lies on the line, then
$x = \frac{2}{3}k \Longleftrightarrow k = \frac{3}{2}x$
substitute $k = -\frac{3}{2}x$ to $y = \frac{4}{3}k = 2x$.
Thus, the locus of point R lies on the line $y = 2x$.
Therefore the problem 6 is analogy to problem 5.

Problem7:
Let $\mathcal{H}$ be the hyperbola $x^2 - y^2 = 1 \cdots (1)$. Prove that, if the stright line $y = 2x + k \cdots (2)$ and $\mathcal{H}$ have two intersections P and Q, the midpoint R of line seqment PQ lies on the line $y = \frac{1}{2}x$ in the part corresponding to $x < -\frac{2\sqrt{3}}{3}$ or $x > \frac{2\sqrt{3}}{3}$.
Solution:

To find the coordinates of P and Q,
substitute (2) to (1), then we obtain

$$\begin{aligned} x^2 - (2x+k)^2 &= 1 \\ x^2 - 4x^2 - 4kx - k^2 &= 1 \\ -3x^2 - 4kx - (k^2+1) &= 0 \end{aligned}$$

Let D be the descriminant of the equation above, then
(1) and (2) have two common points if and only if $D > 0$, then

$$\begin{aligned} \frac{D}{4} = k^2 - 3k^2 - 3 &> 0 \\ k^2 &> 3 \\ k < -\sqrt{3} \quad or \quad k > \sqrt{3} \end{aligned}$$

In order to find the coordinate of R,
put $P(x_1, y_1),\ Q(x_2, y_2),\ R(x, y)$

$$x_1 + x_2 = -\frac{4}{3}k$$
$$y_1 = 2x_1 + k$$
$$y_2 = 2x_2 + k$$
$$y_1 + y_2 = 2(x_1 + x_2) + 2k = -\frac{2}{3}k$$

The locus of R is the midpoint of PQ, then
$R = \dfrac{P+Q}{2} = \left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right) = \left(-\dfrac{2}{3}k, -\dfrac{1}{3}k\right)$
Thus, the coordinate of $R(-\frac{2}{3}k, -\frac{1}{3}k)$
The locus of R lies on the line, then
$x = -\frac{2}{3}k \Longleftrightarrow k = -\frac{3}{2}x$
substitute $k = -\frac{3}{2}x$ to $y = \frac{4}{3}k = \frac{1}{2}x$.
Thus, the locus of point R lies on the line $y = \frac{1}{2}x$ in the part corresponding to $x < -\frac{2\sqrt{3}}{3}$ or $x > \frac{2\sqrt{3}}{3}$.
Therefore the problem 7 is analogy to problem 5.

5.3 Generalization

Problem8:

Take a point A and A' as the points at which the ellipse $\mathcal{E} : \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ intersect the x-axis, and take P and P' as the points at which an arbitrary line $\ell : \ x = x_1$ parallel to the y-axis intersects this ellipse, provided that $x_1 \neq 0$

1) Find the equations of line $A'P$ and $P'A$ by assuming $P(x_1, y_1)$ and $P'(x_1, -y_1)$.

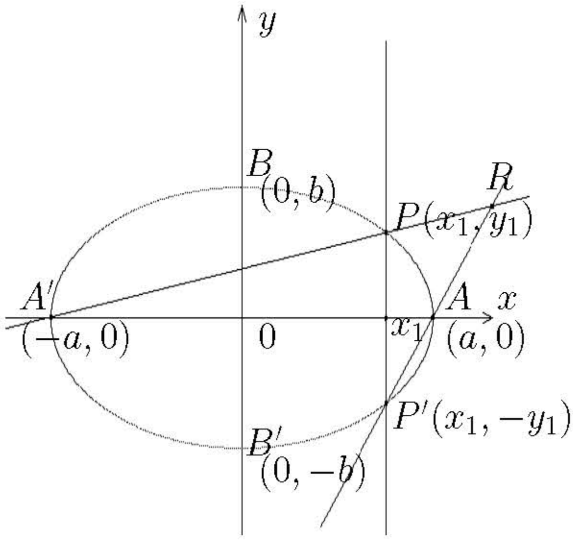
2) Let R be the intersection of the two lines $A'P$ and $P'A$. Prove that the equation of the locus of R is $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

Solution:

1) The intersect point of line: $\ell : x = x_1$ and ellipse are defined by $\mathcal{E} : \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, then the equation of lines

$$A'P : y = \frac{y_1(x+a)}{x_1+a}$$
$$PA' : y = \frac{-y_1(x-a)}{x_1-a}$$

2) Let R be the intersection of $A'P$ and $P'A$
the equation of locus of R we need the equation of the lines:



$$A'P : y = \frac{y_1(x+a)}{x_1+a}$$
$$PA' : y = \frac{-y_1(x-a)}{x_1-a}$$

To get the coordinate of $A'P \cap AP'$, we must solve the system of the equation

$$\frac{y_1(x+a)}{x_1+a} = \frac{-y_1(x-a)}{x_1-a}$$

then we obtain

$$x = \frac{a^2}{x_1}$$
$$y = \frac{ay_1}{x_1}$$

Therefore the coordinate of $R(\frac{a^2}{x_1}, \frac{ay_1}{x_1})$
then $x_1 = \frac{a^2}{x}, \quad y_1 = \frac{ay}{x}$.
Since P is on ellipse, we have

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$$
$$\frac{a^2}{x^2} + \frac{a^2y^2}{b^2x^2} = 1$$
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Therefore the locus of R is hyperbola.

Problem9:

Take a point A and A' as the points at which the hyperbola $\mathcal{H}: \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ intersect the x-axis, and take P and P' as the points at which an arbitrary line $\ell:\ x = x_1$ parallel to the y-axis intersects this hyperbola, provided that $x_1 \neq 0$.

1) Find the equations of line $A'P$ and $P'A$ by assuming $P(x_1, y_1)$ and $P'(x_1, -y_1)$.

2) Let R be the intersection of the two lines $A'P$ and $P'A$. Prove that the equation of the locus of R is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Solution:

1) The intersect point of line: $\ell : x = x_1$ and hyperbola are defined by $\mathcal{H} : \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, then the equation of lines

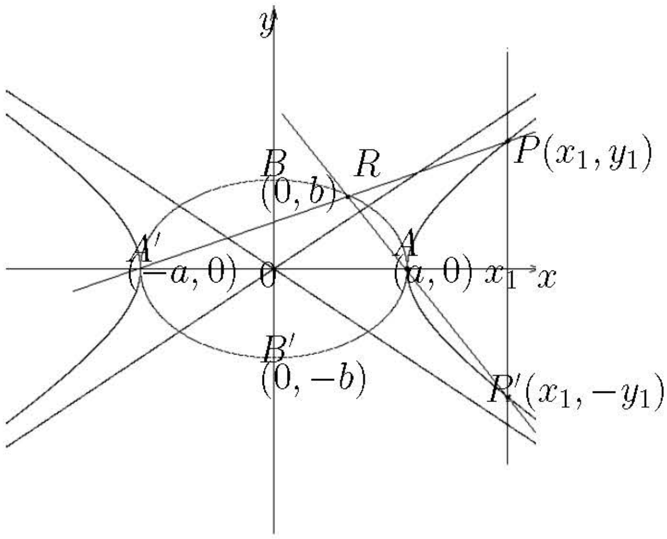
$$A'P : y = \frac{y_1(x+a)}{x_1+a}$$
$$PA' : y = \frac{-y_1(x-a)}{x_1-a}$$

2) Prove the equation of the locus of points R

Let R be the intersection of $A'P$ and $P'A$

Coordinate of $R = A'P \cap AP'$ the equation of locus of R we need the equation of the lines:

$$A'P : y = \frac{y_1(x+a)}{x_1+a}$$
$$PA' : y = \frac{-y_1(x-a)}{x_1-a}$$



To get the coordinate of $A'P \cap AP'$, we must solve the system of the equation

$$\frac{y_1(x+a)}{x_1+a} = \frac{-y_1(x-a)}{x_1-a}$$
$$y_1(x+a)(x_1-a) = -y1(x-a)(x_1+a)$$

When $P' = A'$, then $y_1 = 0$. So y_1 is not appear on curve, then we obtain

$$x = \frac{a^2}{x_1}$$
$$y = \frac{ay_1}{x_1}$$

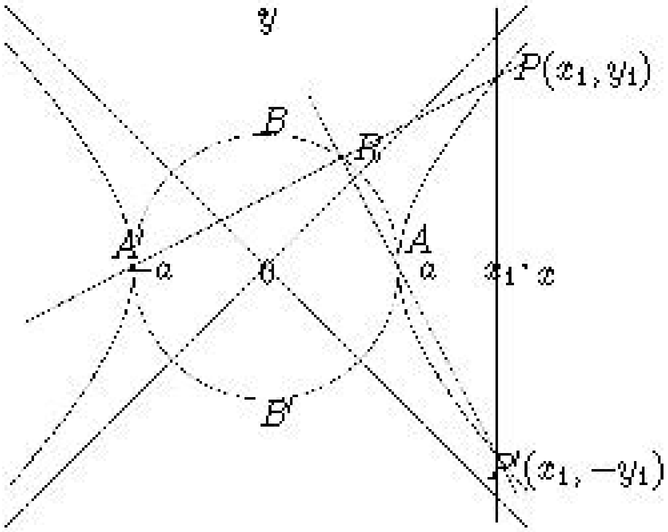
Therefore the coordinate of $R(\frac{a^2}{x_1}, \frac{ay_1}{x_1})$
To obtain the relation between x and y, must eliminate x_1 and y_1
then $x_1 = \frac{a^2}{x}, y_1 = \frac{ay}{x}$.
Since $P(x_1, y_1)$ is on hyperbola, we have

$$\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$$
$$\frac{a^2}{x^2} - \frac{a^2y^2}{b^2x^2} = 1$$
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Therefore the locus of point R is ellipse with two points $(\pm 4, 0)$ removed.

Remark:
Consider special case of problem 6, $a = b$:

Given a rectangular hyperbola $x^2 - y^2 = a^2$, where $a > 0$. Suppose that $x_1 > 0$ or $x_1 < -a$
then, the locus of points R is a part of circle (circle minus two points) with diameter "AA'".



The rectangular hyperbola is the special case of hyperbola just as circle is the special case of ellipse.

Problem10:
Let $\mathcal{E}$ be the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \cdots (1)$. Prove that, if the stright line $y = cx + k \cdots (2)$ and $\mathcal{E}$ have two intersections P and Q, the midpoint R of line seqment PQ lies on the line $y = -cx$ in the part corresponding to $-\sqrt{b^2 + a^2c^2} < x < \sqrt{b^2 + a^2c^2}$.

Solution:
In order to find the coordinates of P and Q
substitute (2) to (1), then

$$\frac{x^2}{a^2} + \frac{(cx+k)^2}{b^2} = 1$$
$$b^2x^2 + a^2(cx+k)^2 = a^2b^2$$
$$b^2x^2 + a^2c^2x^2 + 2a^2akx + a^2k^2 = a^2b^2$$
$$(b^2 + a^2c^2)x^2 + 2a^2ckx + a^2k^2 - a^2b^2 = 0$$

Let D be the discrimenant of the equation above, then
(1) and (2) have two common points if and only if $D > 0$, then

$$\begin{aligned}
\frac{D}{4} &= a^4c^2k^2 - (b^2 + a^2c^2)(a^2k^2 - a^2b^2) > 0 \\
a^4c^2k^2 &- (a^2b^2k^2 - a^2b^4 + a^4c^2k^2 - a^4b^2c^2) > 0 \\
a^4c^2k^2 &- a^2b^2k^2 + a^2b^4 - a^4c^2k^2 + a^4b^2c^2 > 0 \\
&-a^2b^2k^2 + (b^2 + a^2c^2)a^2b^2 > 0 \\
&k^2 < (b^2 + a^2c^2)
\end{aligned}$$

Thus (1) and (2) has two common points when
$\sqrt{b^2 + a^2c^2} < k < \sqrt{b^2 + a^2c^2}$
Put $P(x_1, y_1),\ Q(x_2, y_2),\ R(x, y)$

$$\begin{aligned}
x_1 + x_2 &= -\frac{2a^2ck}{b^2 + a^2c^2} \\
y_1 &= cx_1 + k \\
y_2 &= cx_2 + k \\
y_1 + y_2 &= (x_1 + x_2)c + 2k \\
&= -\frac{2a^2c^2k}{b^2 + a^2c^2} + 2k \\
&= \frac{2b^2}{b^2 + a^2c^2}k
\end{aligned}$$

The locus of R is the midpoint of PQ, then
$R = \frac{P+Q}{2} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(-\frac{a^2c}{b^2 + a2c^2}k, \frac{b^2}{b^2 + a^2c^2}k\right)$
Therefore the locus of $R(-\frac{a^2c}{b^2 + a2c^2}k, \frac{b^2}{b^2 + a^2c^2}k)$
Thus $x = -\frac{a^2c}{b^2 + a2c^2}k, \quad y = \frac{b^2}{b^2 + a^2c^2}k$
In order to find the locus of R lei on the line,
from $x = -\frac{a^2c}{b^2 + a^2c^2}k \Longrightarrow k = -\frac{b^2 + a^2c^2}{a^2c}x$
substitute $k = -\frac{b^2 + a^2c^2}{a^2c}x$ to $y = \frac{b^2}{b^2 + a^2c^2}k$, then
$y = -\frac{b^2}{a^2c}x$

Therefore the locus of point R lies on the line $y = -\frac{b^2}{a^2c}x$ in the part coore-sponding to $-\sqrt{b^2 + a^2c} < x < \sqrt{b^2 + a^2c}$.

Problem11:
Let C be the conic $ax^2+bxy+cy^2+dx+ey+f=0\cdots(1)$ and ℓ be the stright line $y=mx+k\cdots(2)$.
(i) Find the condition on k in terms of a,b,c,d,e,f,m such that ℓ and C have two intersection points.
(ii) Suppose ℓ and C have two intersection points P and Q. Prove that the midpoint R of the line segment PQ lies on some line $y=m'x+k'$. Express m' and k' in terms of m,k,a,b,c,d,e,f.
Solution:
(i) In order to find the intersection of (1) and (2), substitute (2) to (1) then
$ax^2+bx(mx+k)+c(mx+k)^2+dx+ey+f=0$
$ax^2+bmx^2+bkx+cm^2x^2+2cmkx+ck^2+dx+emx+ek+f=0$
$(cm^2+bm+a)x^2+\{(2cm+b)k+(em+d)\}x+(ck^2+ek+f)=0$
Put $A=cm^2+bm+a$. If $A=0$ then the equation is linear, so it has only one solution. So from now on we consider only the case $A\neq 0$.
Let D be the discriminant of the equation above, then

$$\begin{aligned}D&=\{(2cm+b)k+(em+d)\}^2-4(cm^2+bm+a)(ck^2+ek+f)\\&=(b^2-4ac)k^2+2\{(2cd-be)m+bd-2ae\}k\\&\qquad+\{(e^2-4cf)m^2+(2ed-4bf)m+(d^2-4af)\}\end{aligned}$$

Put

$$\begin{aligned}S&=b^2-4ac,\\T&=\{(2cd-be)m+bd-2ae\},\\U&=(e^2-4cf)m^2+(2ed-4bf)m+d^2-4af,\end{aligned}$$

then
$D=Sk^2+2Tk+U$.
Now we consider the condition on k such that (1) and (2) have two common points.
(1) and (2) have two common points if and only if $D>0$.
So we consider the condition on k such that $D>0$.
Case $S=0$
$D=2Tk+U$. So
if $T>0$ then, $D>0\iff k>-\dfrac{U}{2T}$.
if $T<0$ then, $D>0\iff k<-\dfrac{U}{2T}$.

Case $S\neq 0$
This can be found by observing the graph of *quadratic* function $D=Sk^2+$

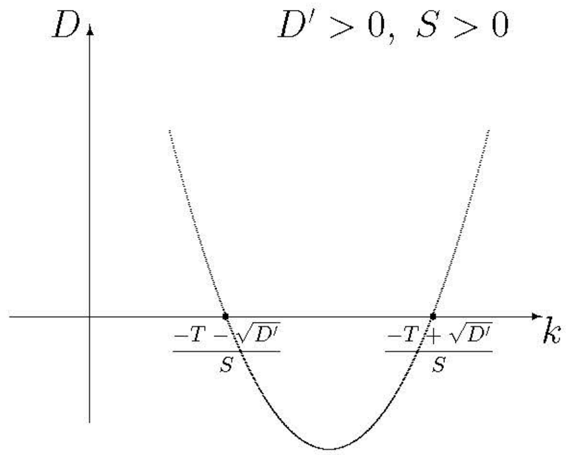
$2Tk + U$ of k.
Let $D' = T^2 - SU$. We know

($\star$) the graph of D and k axis have two common points if and only if $D' > 0$

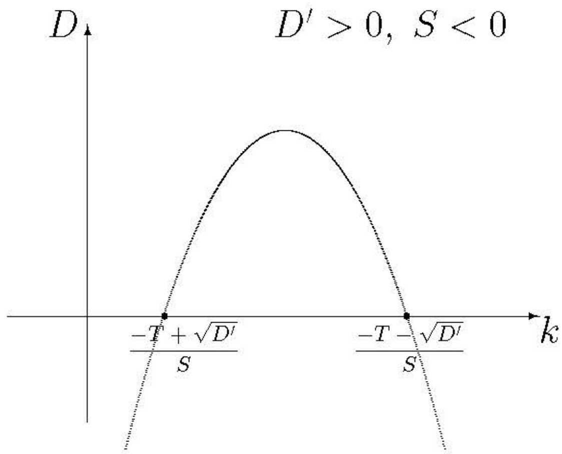
($\star\star$) the graph of D and k axis have exactly one common point if and only if $D' = 0$

($\star\star\star$) the graph of D and k axis have no common point if and only if $D' < 0$.

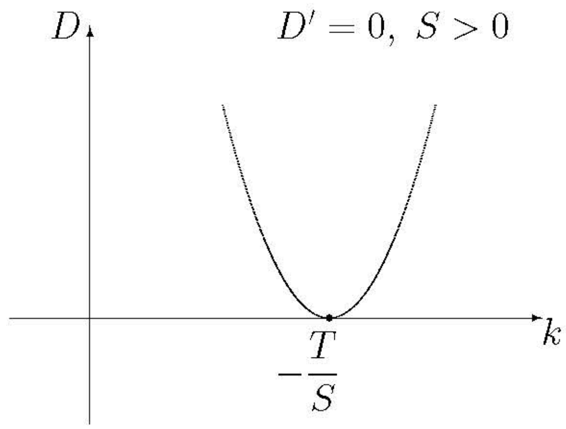
Thus, we obtain the graphs of D as follow:



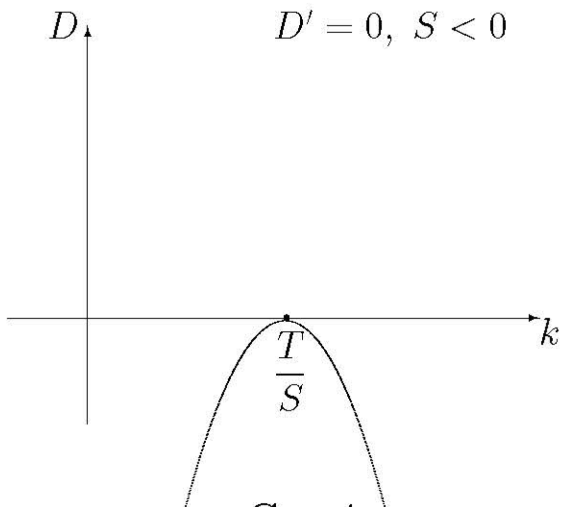
Case1



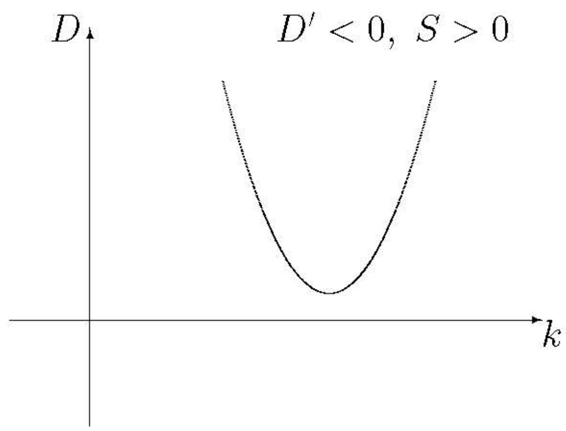
Case2



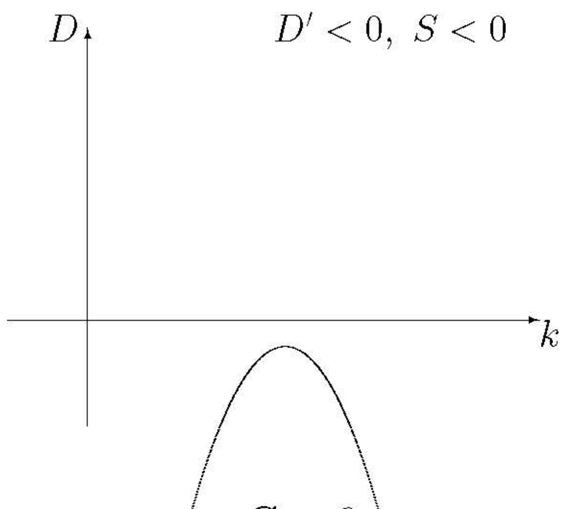
Case3



Case4



Case5



Case6

From the graphs we see that,

If $D' > 0$ and $S > 0$ then, $\quad D > 0 \Longleftrightarrow k < \dfrac{-T-\sqrt{D'}}{S}$ or $k > \dfrac{-T+\sqrt{D'}}{S}$

If $D' > 0$ and $S < 0$ then, $\quad D > 0 \Longleftrightarrow \dfrac{-T-\sqrt{D'}}{S} < k < \dfrac{-T+\sqrt{D'}}{S}$

If $D' = 0$ and $S > 0$ then, $\quad D > 0 \Longleftrightarrow k \neq \dfrac{-T}{S}$

If $D' = 0$ and $S < 0$ then, there is no solution on k

If $D' < 0$ and $S > 0$ then, $\quad D > 0$ for any k

If $D' < 0$ and $S < 0$ then, there is no solution on k

$$
\begin{aligned}
D' &= \{(2cd-be)m+bd-2ae\}^2-(b^2-4ac)\{(e^2-4cf)m^2+(2ed-4bf)m \\
&\quad +d^2-4af)\} \\
&= \{(2cd-be)^2-(b^2-4ac)(e^2-4cf)\}m^2+\{2(2cd-be)(bd-2ae) \\
&\quad -(b^2-4ac)(2ed-4bf)\}m+(bd-2ae)^2-(b^2-4ac)(d^2-4af) \\
&= 4\{(cd-be)cd+(b^2-4ac)cf+ace^2\}m^2+4\{(cd-be)bd+(b^2-4ac)bf \\
&\quad +abe^2\}m+4\{(cd-be)ad+(b^2-4ac)af+a^2e^2\} \\
&= 4\{(cd-be)d+(b^2-4ac)f+ae^2\}\{cm^2+bm+a\}
\end{aligned}
$$

Put $V=(cd-be)d+(b^2-4ac)f+ae^2$, $\;W=cm^2+bm+a$, then
$D' > 0$ if and only if "$V > 0$ and $W > 0$" or "$V < 0$ and $W < 0$" .

(ii) To show that the locus of point R of the line segment PQ lie on some line, we need to find the coordinate of midpoint R

Put $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x, y)$, then

$$
\begin{aligned}
x_1+x_2 &= -\frac{\{(2cm+b)k+(em+d)\}}{cm^2+bm+a} \\
y_1 &= mx_1+k \\
y_2 &= mx_2+k \\
y_1+y_2 &= (x_1+x_2)m+2k \\
&= \frac{(bm+2a)k-em^2+dm}{cm^2+bm+a}
\end{aligned}
$$

Since R is the midpoint of the line segment PQ, we have

$$R = \frac{P+Q}{2} = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$
$$= \left(-\frac{\{(2cm+b)k+(em+d)\}}{2(cm^2+bm+a)}, \frac{(bm+2a)k-em^2-dm}{2(cm^2+bm+a)}\right)$$

Thus, $x = \dfrac{-(2cm+b)k-(em+d)}{2(cm^2+bm+a)}$, $y = \dfrac{(bm+2a)k-em^2-dm}{2(cm^2+bm+a)}$.

Multiply $bm+2a$ to x and $2cm+b$ to y, then add, we obtain

$$(bm+2a)x+(2cm+b)y = \frac{-(em+d)\{(bm+2a)+(2cm+b)m\}}{2(cm^2+bm+a)}$$
$$(bm+2a)x+(2cm+b)y = \frac{-(em+d)\{bm+2a+2cm^2+bm\}}{2(cm^2+bm+a)}$$
$$(bm+2a)x+(2cm+b)y = \frac{-(em+d)\{2cm^2+2bm+2a\}}{2(cm^2+bm+a)}$$
$$(bm+2a)x+(2cm+b)y = \frac{-2(em+d)\{cm^2+bm+a\}}{2(cm^2+bm+a)}$$
$$(bm+2a)x+(2cm+b)y = -(em+d)$$

If $(2cm+b)=0$ and $(bm+2a)\neq 0$, then

$$x = \frac{-(em+d)}{bm+2a}$$

If $(2cm+b)\neq 0$ and $(bm+2a)\neq 0$, then

$$y = \frac{-(em+d)-(bm+2a)x}{2cm+b}$$

Therefore, the midpoint R of the line seqment PQ lies on the line

$$y = \frac{-(em+d)-(bm+2a)x}{2cm+b}$$

Chapter 6

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