A Thesis for the Degree of Ph.D. in Engineering

Analysis of asymptotic forms of solutions of perturbed half-linear ordinary differential equations

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Abstract. Asymptotic forms of nontrivial solutions of half-linear ordinary differential equation

$$\big(|u'|^{\alpha-1}u'\big)' = \alpha\big(1 + b(t)\big)|u|^{\alpha-1}u, \quad \alpha > 0,$$

are investigated under smallness conditions on $b(t)$. It is shown that every nontrivial solution of this equation behaves like ce^{t} or ce^{-t}, $c = const \neq 0$, as $t \to \infty$. When $\alpha = 1$, our results reduce to well-know ones for linear ordinary differential equations.

The proof of one of the main results is based on analysis of solutions of generalized Riccati equations associated with this half-linear equation.

1 Introduction

In modern physical sciences, biological sciences and technology, it is found that various phenomena are described in terms of differential equations. So it is impossible to study modern sciences without deep understanding of differential equations, in particular, nonlinear ordinary differential equations.

Motivated by these facts, the author has decided to devote herself to the study of asymptotic properties of ordinary differential equations. It is *perturbed half-linear equations* that are investigated in this thesis, which are of the form

$$\big(|u'|^{\alpha-1}u'\big)' = \alpha\big(1 + b(t)\big)|u|^{\alpha-1}u. \tag{HL}$$

Here it is assumed that $\alpha > 0$ is a constant, and $b(t)$ is a given continuous function defined near $+\infty$. A C^1-function u defined near $+\infty$ is called a solution of equation (HL) if $|u'|^{\alpha-1}u'$ is of class C^1, and (HL) is satisfied for all sufficiently large t.

When $\alpha = 1$, equation (HL) reduces to the linear equation

$$u'' = (1 + b(t))u. \tag{L}$$

It should be noted that the solution space of (HL) has just one half of the properties which characterize linearity. In fact, if u is a solution of (HL), then so is Cu for any constant $C \in \mathbb{R}$. However, for solutions u_1 and u_2 of

(HL), so is not $u_1 + u_2$, generally. By this fact, equations of the form (HL) are called half-linear equations.

In the recent three decades many similarities between properties of solutions of (HL) and those of (L) were revealed. However, as far as the author knows, some fundamental problems and conjectures concerning asymptotic properties of solutions of (HL) still remain unsolved yet. In this thesis we make an attempt to solve one of such problems. More precisely, we intend to give answers of the following problem:

Problem. *When $b(t)$ is small, in some sense, near $+\infty$ what are the asymptotic forms of solutions of (HL)?*

For the case where $\alpha = 1$, that is, for equation (L) such a problem has been extensively investigated; see Bellman [1], Bodine and Lutz [2], Coppel [3] and Hartman [7]. To get an insight into our problem, let us notice the following two known facts:

Fact 1.1. Let $\int^{\infty} |b(t)|dt < \infty$. Then *linear equation* (L), which is a prototype of (HL), has two independent solutions u_1 and u_2 with the asymptotic forms

$$u_1(t) \sim e^t \text{ and } u_2(t) \sim e^{-t} \text{ as } t \to +\infty,$$

respectively. Since every solution of (L) is expressed as a linear combination of u_1 and u_2, every nontrivial solution u of (L) has the asymptotic form

$$u(t) \sim ce^t \text{ or } u(t) \sim ce^{-t} \text{ as } t \to +\infty,$$

for some constant $c \neq 0$. See for example [1, 3, 7]. (There are many refinements of this property. See [7]). Here, of course, for two positive functions (or negative functions) $f(t)$ and $g(t)$ defined near $+\infty$ the symbol $f(t) \sim g(t)$ means that $\lim_{t\to\infty} f(t)/g(t) = 1$.

Fact 1.2. Let $b(t) \equiv 0$ in (HL), that is, let us consider the simple half-linear equation of constant coefficients

$$\left(|u'|^{\alpha-1}u'\right)' = \alpha|u|^{\alpha-1}u. \tag{HL_0}$$

We can solve this equation explicitly. All of the solutions of (HL_0) are given by

$$ce^t, \quad ce^{-t}, \quad cE(t+t_0), \quad cF(t+t_0)$$

where c and t_0 are constants, E and F are, respectively, the generalized hyperbolic sine function and the generalized hyperbolic cosine function with exponent α. Since

$$E(t) \sim c_1(\alpha)e^t \quad \text{and} \quad F(t) \sim c_2(\alpha)e^t \quad \text{as } t \to \infty,$$

for some constants $c_1(\alpha), c_2(\alpha) > 0$, we find that every nontrivial solution u of (HL_0) satisfies

$$u(t) \sim ce^t \quad \text{or} \quad u(t) \sim ce^{-t} \quad \text{as } t \to \infty,$$

where $c \neq 0$ is a constant. See in detail [4, 6]. (In the Appendix we give the definitions, fundamental properties, and asymptotic properties of $E(t)$ and $F(t)$, and give proofs of them.)

From these facts it is natural to conjecture that, if $b(t)$ is sufficiently small near $+\infty$, then every nontrivial solution u of (HL) has the asymptotic form

$$u(t) \sim ce^t \quad \text{or} \quad u(t) \sim ce^{-t} \quad \text{as } t \to \infty,$$

for some constant $c \neq 0$. In this thesis we give affirmative answers to this conjecture.

This thesis is organized as follows. In Section 2 we collect preparatory results which will be employed latter. In Section 3 we consider our Problem under the signum condition $b(t) \geq 0$ near $+\infty$ or $b(t) \leq 0$ near $+\infty$. We will show that our conjecture is true with these additional conditions. In Section 4 we consider our Problem without signum conditions on $b(t)$. We show that our conjecture is also true if another smallness condition is imposed on $b(t)$.

The difficulty in proving the main results later comes from mainly the following two facts:

(i) The solution space of a half-linear equation is not a linear space;

(ii) There is not a so-called variation of constants formula for half-linear equations.

So in this thesis we must give the proofs of main results without employing the well-known results concerning the properties of linear equations.

2 Preliminaries

In this section we give and prove preparatory results concerning nontrivial solutions of equation (HL).

Firstly we note that local solutions of half-linear equation (HL) and linear equation (L) have the following common property:

Lemma 2.1. *Every initial value problem of equation (HL) has the unique solution which exists globally on the whole interval under consideration. Therefore, solutions $u(t)$ of (HL) satisfying $u(t_0) = u'(t_0) = 0$ for some t_0 must be $u(t) \equiv 0$.*

See [4, Theorem 1.1.1] or [10, Lemma 4.2] for the proof of this lemma.

Lemma 2.2. *Let $1 + b(t) \geq 0$ for sufficiently large t. Then every nontrivial solution of (HL) is of constant sign near $+\infty$. (So it has no zeros near $+\infty$.)*

Proof. Let $T > 0$ be a sufficiently large number such that $1 + b(t) \geq 0$ for $t \geq T$, and $u(t)$ be a nontrivial solution of (HL) on $[T, \infty)$. If u is not of constant sign near $+\infty$, then there are two points $T_1, T_2 \geq T$ satisfying $T_1 < T_2$ and

$$u(T_1) = 0, u'(T_1) > 0, u'(T_2) = 0, \text{ and } u(t) > 0 \text{ for } t \in (T_1, T_2).$$

(By Lemma 2.1, the case that $u'(T_1) = 0$ is excluded.) An integration of (HL) on $[T_1, T_2]$ gives

$$-\big[u'(T_1)\big]^\alpha = \alpha \int_{T_1}^{T_2} \big(1 + b(s)\big) u(s)^\alpha ds,$$

which is an obvious contradiction. So u is of constant sign near $+\infty$. This completes the proof. □

Lemma 2.3. *Let $1 + b(t) \geq 0$ for sufficiently large t and $\int^\infty |b(t)| dt < \infty$. Then every nontrivial solution u of equation (HL) satisfies one of the following two properties as $t \to \infty$:*

(i) $|u'(t)| \uparrow \infty$ (and therefore $|u(t)| \uparrow \infty$) as $t \to \infty$;

(ii) $|u'(t)| \downarrow 0$ and $|u(t)| \downarrow 0$ as $t \to \infty$.

Since $u(t)$ is a solution of (HL) if and only if so is $-u(t)$, by Lemma 2.2 we may assume that $u(t) > 0$ near $+\infty$. Therefore in this thesis below we will consider mainly (eventually) positive solutions of equation (HL). Note that, for positive solutions u of (HL), the properties (i) and (ii) of Lemma 2.3 can be recasted as follows:

(i) $u'(t) \uparrow \infty$ (and therefore $u(t) \uparrow \infty$) as $t \to \infty$;

(ii) $u'(t) \uparrow 0$, and $u(t) \downarrow 0$ as $t \to \infty$.

For simplicity let us call positive solutions u of (HL) (or of equations of the same types) satisfying the property (i) as *increasing solutions,* and positive solutions u satisfying the property (ii) as *decreasing solutions,* respectively.

Proof of Lemma 2.3. Let $T > 0$ be a sufficiently large number such that $1 + b(t) \geq 0,\ t \geq T$. We may suppose that $u(t) > 0,\ t \geq T$. Then, by equation (HL) we see that $|u'(t)|^{\alpha-1}u'(t)$ is increasing on $[T, \infty)$; that is, $u'(t)$ is increasing on $[T, \infty)$. We divide the argument into several cases by the limit of $u'(t)$ as $t \to \infty$.

Let $u'(t) \uparrow \infty$ as $t \to \infty$. Then the property (i) of the statement holds.

Next, let $u'(t) \uparrow c$ as $t \to \infty$ for some constant $c > 0$. Then $u(t) \sim ct$ as $t \to \infty$, and an integration of equation (HL) gives

$$
\begin{aligned}
|u'(t)|^{\alpha-1}u'(t) - |u'(T)|^{\alpha-1}u'(T) &= \alpha \int_T^t \big(1 + b(s)\big)u(s)^\alpha ds \\
&\geq c_1 \int_T^t \big(1 + b(s)\big)s^\alpha ds \qquad (2.1)
\end{aligned}
$$

for some constant $c_1 > 0$. Since

$$
\begin{aligned}
\int_T^t \big(1 + b(s)\big)s^\alpha ds &\geq \frac{1}{\alpha+1}\big(t^{\alpha+1} - T^{\alpha+1}\big) - t^\alpha \int_T^\infty |b(s)|ds \\
&\longrightarrow \infty \text{ as } t \to \infty,
\end{aligned}
$$

(2.1) is a contradiction to the fact $\lim_{t\to\infty} u'(t) = c$.

Let $u'(t) \uparrow 0$ as $t \to \infty$. Then $u'(t) < 0$ near $+\infty$; and therefore $u(t)$ decreases near $+\infty$. Since $u(t) > 0$, we have $u(t) \downarrow l$ for some constant $l \geq 0$. If $l > 0$, then $u(t) \geq l$ near $+\infty$. We get from (2.1)

$$
\begin{aligned}
|u'(t)|^{\alpha-1}u'(t) - |u'(T)|^{\alpha-1}u'(T) &\geq l^\alpha \int_T^t \big(1 + b(s)\big)ds \\
&\geq l^\alpha \Big\{(t - T) - \int_T^\infty |b(s)|ds\Big\} \\
&\longrightarrow \infty \text{ as } t \to \infty.
\end{aligned}
$$

This is a contradiction to the fact $\lim_{t\to\infty} u'(t) = 0$. Therefore $u(t) \downarrow 0$ (and $u'(t) \uparrow 0$), and the property (ii) holds.

Finally let $u'(t) \uparrow c$ as $t \to \infty$ for some constant $c < 0$. However, this implies that $u(t) \sim ct$ as $t \to \infty$. Since $u(t) > 0$, this is an obvious contradiction.

This completes the proof. □

Remark 2.4. Under the assumption of Lemma 2.3, positive solutions u of (HL) satisfying the property (i) and the property (ii) of Lemma 2.3, respectively, surely exist.

In fact, let u be the solution of (HL) with the initial condition $u(t_0) = u_0 > 0$ and $u'(t_0) = u_1 > 0,\ t_0 > 0$. Then we can show that $u'(t)$ remains positive as long as u exists. Since every local solution of (HL) can be prolonged to $+\infty$ [4, Theorem 1.1.1], this $u(t)$ satisfies $u(t), u'(t) > 0$ for $t \geq t_0$; and so $u(t)$ satifies the property (i) of Lemma 2.3.

The existence of solutions satisfying the property (ii) of Lemma 2.3 was proved in [10, Theorem 5.1].

3 The case where $b(t)$ is of constant signs

In this section we give the affirmative answer to our conjecture under the condition that $b(t)$ is of constant sign near $+\infty$. Thus, it is convenient to rewrite equation (HL) in the following two equations:

$$\left(|u'|^{\alpha-1}u'\right)' = \alpha(1 + p(t))|u|^{\alpha-1}u, \qquad (HL_+)$$

$$\left(|u'|^{\alpha-1}u'\right)' = \alpha(1 - p(t))|u|^{\alpha-1}u. \qquad (HL_-)$$

In this section we assume the next conditions:

(A_1) $p \in C[0, \infty)$;
(A_2) $p(t) \geq 0$ near $+\infty$ for (HL_+); $0 \leq p(t) \leq 1$ near $+\infty$ for (HL_-),
(A_3) $\int^\infty p(t)dt < \infty$.

The following is the main result of this section which gives an answer to our Problem:

Theorem 3.1. *([8]) Under assumptions $(A_1) - (A_3)$, every nontrivial solution u of (HL_+) and (HL_-) has the asymptotic form*

$$u(t) \sim ce^t \qquad \text{or} \qquad u(t) \sim ce^{-t} \quad \text{as } t \to \infty, \qquad (3.1)$$

for some constant $c \neq 0$.

More precisely, every nontrivial solution u of $(HL_\pm)$ satisfying the property (i) of Lemma 2.3 has the former asymptotic form of (3.1), and every nontrivial solution u of $(HL_\pm)$ satisfying the property (ii) of Lemma 2.3 has the latter.

To give the proof of Theorem 3.1 we prepare further several lemmas. The following simple point-wise inequalities are used to estimate several integrals in the sequel:

Lemma 3.2. (i) *Let $\beta \geq 1$. Then $(1-x)^\beta \geq 1 - \beta x$ for $x \in [0,1]$.*

(ii) *Let $0 < \beta \leq 1$. Then $(1-x)^\beta \geq 1 - x$ for $x \in [0,1]$.*

(iii) *Let $0 < \beta \leq 1$. Then $(1+x)^\beta \leq 1 + x$ for $x \geq 0$.*

(iv) *Let $\beta \geq 1$ and $M > 0$ be a constant. Then there is a constant $K = K_M > 0$ such that*

$$(1+x)^\beta \leq 1 + Kx \quad \text{for } x \in [0, M].$$

(In fact, we may take $K = [(1+M)^\beta - 1]/M$.)

The following comparison principle will be employed in several places. The proof is found, for example, in [10, Lemma 4.1].

Lemma 3.3. *Suppose that $p_1, p_2 \in C[t_0, t_1]$ and $0 \leq p_1(t) \leq p_2(t)$ on $[t_0, t_1]$. Let u_i, $i = 1, 2$, be solutions on $[t_0, t_1]$ of the equations*

$$(|u_i'|^{\alpha-1} u_i)' = p_i(t)|u_i|^{\alpha-1}u_i, \ i = 1, 2,$$

respectively, satisfying

$$u_1(t_0) \leq u_2(t_0) \text{ and } u_1'(t_0) < u_2'(t_0).$$

Then $u_1(t) < u_2(t)$ and $u_1'(t) < u_2'(t)$ on $(t_0, t_1]$.

Since conditions (A_2) and (A_3) are assumed throughout this section (especially in Theorem 3.1), every positive solution of equation (HL_+) and (HL_-) satisfies the property either (i) or (ii) of Lemma 2.3. We will consider asymptotic forms of positive solutions of these two types separately: in Section 3.1 we give the asymptotic forms of positive solutions satisfying the property (i) of Lemma 2.3; that is, positive increasing solutions; and in Section 3.2 we do so for positive decreasing solutions. The proof of Theorem 3.1 will be completed by unifying these results in Sections 3.1 and 3.2.

3.1 Asymptotic forms of positive increasing solutions of $(HL_\pm)$.

Here, as stated above, we treat positive increasing solutions of $(HL_\pm)$; that is positive solutions u of $(HL_\pm)$ satisfying the property (i) of Lemma 2.3: $u'(t) \uparrow \infty$ and $u(t) \uparrow \infty$ as $t \to \infty$.

Lemma 3.4. (i) Let u be a positive solution of (HL_+) on $[T, \infty)$ satisfying the property (i) of Lemma 2.3 for sufficiently large $T > 0$. Then

$$u(t) \geq ce^t, \quad t \geq T, \text{ for some constant } c > 0. \tag{3.2}$$

(ii) Let u be a positive solution of (HL_-) on $[T, \infty)$ satisfying the property (i) of Lemma 2.3 for sufficiently large $T > 0$. Then

$$u(t) \leq ce^t, \quad t \geq T, \text{ for some constant } c > 0. \tag{3.3}$$

Proof. We give only the proof of (i), because (ii) can be proved similarly.

We may assume that $u'(t) > 0$ on $[T, \infty)$. Let $c > 0$ be a sufficiently small number such that

$$u(T) > ce^T \text{ and } u'(T) > ce^T.$$

Put $z(t) = ce^t, t \geq T$. Then z satisfies $u(T) > z(T),\ u'(T) > z'(T)$, and

$$(|z'|^{\alpha-1}z')' = \alpha|z|^{\alpha-1}z,\ t \geq T.$$

By Lemma 3.3 we obtain (3.2) as desired. □

Lemma 3.5. *Let u be a positive solution of (HL_+) or (HL_-) satisfying the property (i) of Lemma 2.3. Then the function $u(t)/e^t$ is eventually monotone near $+\infty$.*

Proof. Let $u(t), u'(t) > 0$ on $[T, \infty)$ and put $v(t) = u(t)/e^t$. We will show that $v'(t) \geq 0$ near $+\infty$, or $v'(t) \leq 0$ near $+\infty$, by contradiction.

If this is not the case, then there are three points t_1, t_2 and t_3 $(T < t_1 < t_2 < t_3)$ satisfying

$$v'(t_1)v'(t_2) < 0 \text{ and } v'(t_1)v'(t_3) > 0.$$

We can assume that

$$v'(t_1) > 0,\ v'(t_2) < 0,\ \text{and } v'(t_3) > 0.$$

Then there are two points $\tau_1 \in (t_1, t_2)$ and $\tau_2 \in (t_2, t_3)$ such that

$$\begin{aligned} &v'(\tau_1) = 0, \qquad v''(\tau_1) \le 0, \qquad \text{and} \\ &v'(\tau_2) = 0, \qquad v''(\tau_2) \ge 0. \end{aligned} \tag{3.4}$$

On the other hand, note that $v(t)$ satisfies $e^{-t}u' = v + v' > 0,\ t \ge T$, and

$$\Big[\big(v+v'\big)^{\alpha}\Big]' + \alpha\Big(v+v'\Big)^{\alpha} = \alpha\Big(1 \pm p(t)\Big)v^{\alpha};$$

that is

$$v'' + 2v' + v = \Big(1 \pm p(t)\Big)\Big(v+v'\Big)^{1-\alpha} v^{\alpha}. \tag{3.5}$$

Let us divide the proof into two cases.

Case 1. The case where $p(t) > 0,\ t \in [T, t_3]$. Since $v'(\tau_i) = 0$, by equation (3.5), $v''(\tau_i) = \pm p(\tau_i)v(\tau_i), i = 1, 2$. So $v''(\tau_1)$ and $v''(\tau_2)$ have the same signs, which is an obvious contradiction to the properties (3.4).

Case 2. The case where $p(t) \ge 0,\ t \in [T, t_3]$. Let $\{p_\varepsilon(t)\}_{\varepsilon>0}$ be a family of continuous functions of $(t, \varepsilon) \in [T, t_3] \times (0, \varepsilon_0], \varepsilon_0 = \text{const} > 0$, satisfying

$$p_\varepsilon(t) > p(t) \text{ on } [T, t_3], \text{ and } \lim_{\varepsilon\to+0}\Big(\max_{[T,t_3]}\big(p_\varepsilon(t) - p(t)\big)\Big) = 0.$$

Further, let $z = z_\varepsilon$ be the solution of the initial value problem

$$\begin{cases} z'' + 2z' + z = \big(1 \pm p_\varepsilon(t)\big)(z+z')^{1-\alpha}z^{\alpha}, \\ z(T) = v(T),\ z'(T) = v'(T). \end{cases} \tag{3.6}$$

By the continuous dependence on the parameter [4, 11], for sufficiently small $\varepsilon > 0, z = z_\varepsilon(t)$ exists at least for $t \in [T, t_3],\ z(t) > 0,\ z(t) + z'(t) > 0$ for $t \in [T, t_3]$, and

$$\lim_{\varepsilon\to+0}\Big(\max_{[T,t_3]}\big|z'_\varepsilon(t) - v'(t)\big|\Big) = 0.$$

Let $m > 0$ be a sufficiently small number satisfying

$$v'(t_1) > m > 0,\ v'(t_2) < -m < 0 \text{ and } v'(t_3) > m > 0.$$

For sufficiently small $\varepsilon > 0$, we have

$$|z'_\varepsilon(t) - v'(t)| < m/2 \qquad \text{for} \quad t \in [T, t_3],$$

which implies that

$$z_\varepsilon'(t_1) > v'(t_1) - (m/2) > m/2 > 0,$$
$$z_\varepsilon'(t_2) < v'(t_2) + (m/2) < -m/2 < 0, \quad \text{and}$$
$$z_\varepsilon'(t_3) > v'(t_3) - (m/2) > m/2 > 0.$$

By noting that $z = z_\varepsilon$ satisfies equation (3.6) and $p_\varepsilon(t) > 0$ on $[T, t_3]$, we find that this is a contradiction as in Case 1.

The proof is complete. □

Proposition 3.6. *Every positive increasing solution u of equations (HL_+) and (HL_-) has the asymptotic form*

$$u(t) \sim ce^t \text{ as } t \to \infty \text{ for some constant } c > 0. \tag{3.7}$$

Proof of Proposition 3.6 for (HL_+). By Lemma 3.5 the function $u(t)/e^t$ is monotone near $+\infty$. If $u(t)/e^t$ decreases, then by (i) of Lemma 3.4 we find that $u(t)/e^t$ decreases to a positive constant as $t \to \infty$; and so (3.7) holds as desired.

Next let $u(t)/e^t$ increase near $+\infty$. We may suppose that $u' > 0$ and $u(t)/e^t$ increases on $[T, \infty)$. An integration of both sides of (HL_+) on $[T, t]$ gives

$$u'(t)^\alpha = u'(T)^\alpha + \alpha \int_T^t (1 + p(s))u(s)^\alpha ds.$$

Since $u(t)/e^t$ increases, we get from the above

$$\begin{aligned} u'(t)^\alpha &\le u'(T)^\alpha + \alpha \frac{u(t)^\alpha}{e^{\alpha t}} \int_T^t (e^{\alpha s} + e^{\alpha s} p(s))ds \\ &= u'(T)^\alpha + \alpha \frac{u(t)^\alpha}{e^{\alpha t}} \Big[\frac{1}{\alpha}(e^{\alpha t} - e^{\alpha T}) + \int_T^t e^{\alpha s} p(s)ds\Big]. \end{aligned}$$

Thus we obtain

$$u'(t)^\alpha \le u'(T)^\alpha + u(t)^\alpha \Big[1 + \alpha e^{-\alpha t} \int_T^t e^{\alpha s} p(s)ds\Big]. \tag{3.8}$$

The computation below slightly differs according to the value of α.

Firstly, let $\alpha > 1$. By the simple inequality

$$(X + Y)^{1/\alpha} \le X^{1/\alpha} + Y^{1/\alpha} \text{ for } X, Y \ge 0,$$

we get from (3.8)

$$u'(t) \le u'(T) + u(t)\Big[1 + \alpha e^{-\alpha t}\int_T^t e^{\alpha s}p(s)ds\Big]^{1/\alpha}.$$

Further, by (iii) of Lemma 3.2 we have

$$u'(t) \le u'(T) + u(t)\Big[1 + \alpha e^{-\alpha t}\int_T^t e^{\alpha s}p(s)ds\Big].$$

By (i) of Lemma 3.4 we obtain

$$\frac{u'(t)}{u(t)} \le c_1 e^{-t} + 1 + \alpha e^{-\alpha t}\int_T^t e^{\alpha s}p(s)ds,$$

for some constant $c_1 > 0$. An integration of both sides gives

$$\log\frac{u(t)}{u(T)} \le (t-T) + c_1\int_T^t e^{-s}ds + \alpha\int_T^t \Big(e^{-\alpha s}\int_T^s e^{\alpha r}p(r)dr\Big)ds.$$

Since

$$\begin{aligned}\int_T^t \Big(e^{-\alpha s}\int_T^s e^{\alpha r}p(r)dr\Big)ds &= \frac{1}{\alpha}\int_T^t p(s)\big(1-e^{-\alpha(t-s)}\big)ds\\ &\le \frac{1}{\alpha}\int_T^\infty p(s)ds < \infty,\end{aligned}$$

we can get

$$\log\frac{u(t)}{u(T)} \le t + O(1), \qquad \text{as } t\to\infty,$$

which implies that $u(t) = O(e^t)$ as $t\to\infty$. By recalling the assumption that $u(t)/e^t$ increases, we find that (3.7) holds.

Next let $0 < \alpha < 1$. From (3.8) we have

$$\begin{aligned}u'(t) &\le u(t)\Big[1 + \frac{u'(T)^\alpha}{u(t)^\alpha} + \alpha e^{-\alpha t}\int_T^t e^{\alpha s}p(s)ds\Big]^{1/\alpha}\\ &\equiv u(t)\big(1+B(t)\big)^{1/\alpha}.\end{aligned}$$

Here $B(t)$ is defined naturally by the last equality. Since $u(t)/e^t$ increases, we find for some constants c_2 and $c_3 > 0$

$$
\begin{aligned}
0 \le B(t) &\le c_2 e^{-\alpha t} + \alpha e^{-\alpha t} \cdot e^{\alpha t} \int_T^t p(s)ds \\
&\le c_3 + \alpha \int_T^\infty p(s)ds < \infty.
\end{aligned}
$$

Therefore by (iv) of Lemma 3.2 we obtain for some constant $K > 0$

$$
u'(t) \le u(t)\Big[1 + \frac{Ku'(T)^\alpha}{u(t)^\alpha} + K\alpha e^{-\alpha t} \int_T^t e^{\alpha s} p(s)ds\Big].
$$

Dividing the both sides by $u(t)$, and integrating on $[T, t]$, we have

$$
\begin{aligned}
\log \frac{u(t)}{u(T)} &\le t - T + c_2 \int_T^t e^{-\alpha s} ds + K\alpha \int_T^t \Big(e^{-\alpha s} \int_T^s e^{\alpha r} p(r)dr\Big)ds \\
&\le t + O(1) + K \int_T^\infty p(s)ds.
\end{aligned}
$$

as $t \to \infty$. So $u(t) = O(e^t)$ as $t \to \infty$, which implies that (3.7) holds as before. This completes the proof. □

Proof of Proposition 3.6. for (HL_-). The argument here is parallel to that in the proof of Proposition 3.6 for (HL_+).

By Lemma 3.5 the function $u(t)/e^t$ is monotone near $+\infty$. If $u(t)/e^t$ increases, then by (ii) of Lemma 3.4 we find that $u(t)/e^t$ increases to a positive constant as $t \to \infty$; and so (3.7) holds as desired.

Next let $u(t)/e^t$ decrease near $+\infty$. We may suppose that $u' > 0$ and $u(t)/e^t$ decreases on $[T, \infty)$. An integration of both sides of (HL_-) on $[T, t]$ gives

$$
u'(t)^\alpha = u'(T)^\alpha + \alpha \int_T^t (1 - p(s))u(s)^\alpha ds.
$$

Employing the decreasing property of $u(t)/e^t$, we get

$$\begin{aligned} u'(t)^\alpha &\geq \alpha \int_T^t (1-p(s)) e^{\alpha s} \Big[\frac{u(s)}{e^s}\Big]^\alpha ds \\ &\geq \alpha \frac{u(t)^\alpha}{e^{\alpha t}} \int_T^t \Big(e^{\alpha s} - e^{\alpha s} p(s)\Big) ds \\ &= u(t)^\alpha \Big[1 - \Big(e^{-\alpha(t-T)} + \alpha e^{-\alpha t} \int_T^t e^{\alpha s} p(s) ds\Big)\Big] \qquad (3.9) \\ &\equiv u(t)^\alpha \big(1 - B(t)\big). \end{aligned}$$

Here $p(s) \leq 1$ of course, and $B(t)$ is defined naturally by the last equality. Since $0 \leq p(s) \leq 1$ we observe that

$$0 \leq B(t) \leq e^{-\alpha(t-T)} + \alpha e^{-\alpha t} \int_T^t e^{\alpha s} ds = 1, \qquad \text{for } t \geq T.$$

So by (i) and (ii) of Lemma 3.2 we obtain from (3.9)

$$u'(t) \geq u(t) \Big[1 - c\Big(e^{-\alpha(t-T)} + \alpha e^{-\alpha t} \int_T^t e^{\alpha s} p(s) ds\Big)\Big], \qquad (3.10)$$

where $c > 0$ is a constant given by

$$c = \begin{cases} 1/\alpha & \text{if } \ 0 < \alpha < 1; \\ 1 & \text{if } \ \alpha > 1. \end{cases}$$

As before, we get from (3.10)

$$\begin{aligned} \log \frac{u(t)}{u(T)} &\geq t - T - c \int_T^t e^{-\alpha(s-T)} ds - c\alpha \int_T^t e^{-\alpha s} \int_T^s e^{\alpha r} p(r) dr ds \\ &= t + O(1) \qquad \text{as} \quad t \to \infty. \end{aligned}$$

So, $u(t)/e^t \geq c_4 > 0$ for some constant c_4, and we find that (3.7) holds.

This completes the proof. □

3.2 Asymptotic forms of positive decreasing solutions of $(HL_\pm)$.

In this subsection we treat positive decreasing solutions u of $(HL_\pm)$; that is, positive solutions u of $(HL_\pm)$ satisfying the property (ii) of Lemma 2.3: $u(t) \downarrow 0$ and $u'(t) \uparrow 0$ as $t \to \infty$.

To state auxiliary results which will be employed in proving the main result, let us consider two half-linear equations of the form $(HL_\pm)$ for a moment:

$$(|W'|^{\beta-1}W')' = Q(t)|W|^{\beta-1}W, \quad t \geq 0; \tag{A_Q}$$

$$(|w'|^{\beta-1}w')' = q(t)|w|^{\beta-1}w, \quad t \geq 0. \tag{A_q}$$

Here we assume the following:
(i) $\beta > 0$ is a constant; and $Q, q \in C[0, \infty)$;
(ii) $Q(t) \geq q(t) > 0,\ t \geq 0$;
(iii) $\int^\infty q(t)dt = \infty$ (therefore, $\int^\infty Q(t)dt = \infty$).

Let $T \geq 0$ and $h > 0$ be arbitrary numbers. Then, by [10, Theorem 5.1], equations (A_Q) and (A_q), respectively, have unique positive solutions $W(t)$ and $w(t)$ on $[T, \infty)$ satisfying

$$W(T) = h,\ W(t) \downarrow 0 \quad \text{and} \quad W'(t) \uparrow 0 \quad \text{as} \quad t \to \infty;$$

and

$$w(T) = h,\ w(t) \downarrow 0 \quad \text{and} \quad w'(t) \uparrow 0 \quad \text{as} \quad t \to \infty.$$

Such solutions are often called *Kneser solutions or positive decaying solutions.* Note that positive solutions of $(HL_\pm)$ satisfying the property (ii) of Lemma 2.3 are positive decaying solutions of $(HL_\pm)$.

For example, the positive decaying solution u of the equation

$$(|u'|^{\beta-1}u')' = \beta|u|^{\beta-1}u, \qquad t \geq 0.$$

passing through the point (T, h) in the $tu-$ plane is given by $u(t) = he^{-(t-T)}$.

The following comparison lemma concerning positive decaying solutions of equations (A_Q) and (A_q) plays an important role to prove our main results.

Lemma 3.7. *Let W and w be positive decaying solutions of equation (A_Q) and (A_q) on $[T, \infty)$, respectively, passing through the point (T, h), $T \geq 0, h > 0$. Then, $W(t) \leq w(t)$ for $t > T$.*

Proof. The proof is done by contradiction. Suppose the contrary that $W(t) > w(t)$ for some $t > T$. Then we can find an interval $[t_0, t_1] \subset [T, \infty)$ such that

$$W(t_0) = w(t_0), \text{ and } W(t) > w(t), \quad \text{in } (t_0, t_1]. \tag{3.11}$$

We claim that $W'(\tau) > w'(\tau)$ for some $\tau \in [t_0, t_1]$. For, if there are no such points, that is, if $W'(t) \le w'(t)$ on $[t_0, t_1]$, then the function $W(t) - w(t)$ is nonincreasing on $[t_0, t_1]$. So $W(t) - w(t) \le W(t_0) - w(t_0) = 0$. However this contradicts to (3.11). Hence $W'(\tau) > w'(\tau)$ for some $\tau \in [t_0, t_1]$.

Since $W(\tau) > w(\tau)$, Lemma 3.3 implies that $W(t) > w(t)$ for $t \ge \tau$. From (A_Q) and (A_q) we obtain

$$\begin{aligned} |W'(t)|^{\beta-1}W'(t) - |w'(t)|^{\beta}w'(t) \\ &= |W'(\tau)|^{\beta-1}W'(\tau) - |w'(\tau)|^{\beta-1}w'(\tau) \\ &+ \int_\tau^t \Big[Q(s)W(s)^\beta - q(s)w(s)^\beta\Big]ds \\ &> |W'(\tau)|^{\beta-1}W'(\tau) - |w'(\tau)|^{\beta-1}w'(\tau), \qquad \text{for} \quad t \ge \tau. \end{aligned}$$

Since $\lim_{t\to\infty} W'(t) = \lim_{t\to\infty} w'(t) = 0$, by letting $t \to \infty$ we obtain

$$0 \ge |W'(\tau)|^{\beta-1}W'(\tau) - |w'(\tau)|^{\beta-1}w'(\tau) > 0.$$

This is a contradiction to the definition of τ. This completes the proof. □

Lemma 3.8. (i) Let u be a positive solution of equation (HL_+) on $[T, \infty)$ satisfying the property (ii) of Lemma 2.3 for sufficiently large $T > 0$. Then

$$u(t) \le ce^{-t}, \quad t \ge T, \text{ for some constant } c > 0. \tag{3.12}$$

(ii) Let u be a positive solution of equation (HL_-) on $[T, \infty)$ satisfying the property (ii) of Lemma 2.3 for sufficiently large $T > 0$. Then

$$u(t) \ge ce^{-t}, \quad t \ge T, \text{ for some constant } c > 0. \tag{3.13}$$

Proof. We give only the proof of (i), because (ii) can be proved similarly.

Let $z(t)$ be the positive decaying solution of equation

$$\Big(|z'|^{\alpha-1}z'\Big)' = \alpha|z|^{\alpha-1}z,$$

passing through the point $(T, u(T))$; that is, $z(t) = u(T)e^{-(t-T)}$. Since $\alpha \le \alpha\big(1 + p(t)\big)$, Lemma 3.7 implies that

$$u(t) \le z(t) \equiv u(T)e^{-(t-T)}, \quad t \ge T,$$

which show that (3.12) holds. This completes the proof. □

Lemma 3.9. *Let u be a positive solution of (HL_-) or (HL_+) satisfying the property (ii) of Lemma 2.3. Then the function $u(t)/e^{-t}$ is eventually monotone.*

Proof. Put $v = u(t)/e^{-t}$, Then $v - v' > 0$ and v satisfies

$$v'' - 2v' + v = \big(1 \pm p(t)\big)(v - v')^{1-\alpha}v^{\alpha},$$

for large t. If $v'(\tilde{t}) = 0$ for some sufficiently large $\tilde{t}$, then $v''(\tilde{t}) = \pm p(\tilde{t})v(\tilde{t})$. So arguing as in the proof of Lemma 3.5, we find that $u(t)/e^{-t}(\equiv v(t))$ is eventually monotone. This completes the proof. □

Proposition 3.10. *Every positive decreasing solution u of equations (HL_+) and (HL_-) has the asymptotic form*

$$u(t) \sim ce^{-t} \text{ as } t \to \infty \text{ for some constant } c > 0. \tag{3.14}$$

Proof of Proposition 3.10 for (HL_+). By Lemma 3.9 the function $u(t)/e^{-t}$ is eventually monotone. If $u(t)/e^{-t}$ increases, then by (i) of Lemma 3.8 we find that $u(t)/e^{-t}$ converges to a positive constant as $t \to \infty$; so (3.14) holds.

Next let $u(t)/e^{-t}$ decrease near $+\infty$. We may suppose that $u' < 0$ and $u(t)/e^{-t}$ decreases on $[T, \infty)$. Since $u'(\infty) = 0$, from (HL_+) we have

$$\big[-u'(t)\big]^{\alpha} = \alpha \int_t^{\infty} \big(1 + p(s)\big)u(s)^{\alpha}ds.$$

The monotonicity of $e^t u(t)$ implies that

$$\begin{aligned}[-u'(t)]^{\alpha} &= \alpha \int_t^{\infty} e^{-\alpha s}\big(1 + p(s)\big)\big[e^s u(s)\big]^{\alpha}ds \\ &\le \alpha e^{\alpha t}u(t)^{\alpha} \int_t^{\infty} e^{-\alpha s}\big(1 + p(s)\big)ds.\end{aligned}$$

Thus

$$-u'(t) \le u(t)\Big(1 + \alpha e^{\alpha t}\int_t^{\infty} e^{-\alpha s}p(s)ds\Big)^{1/\alpha}.$$

Firstly let $\alpha > 1$. Then by (iii) of Lemma 3.2 we obtain

$$-u'(t) \le u(t)\Big(1 + \alpha e^{\alpha t}\int_t^{\infty} e^{-\alpha s}p(s)ds\Big), \tag{3.15}$$

that is

$$-\frac{u'(t)}{u(t)} \le 1 + \alpha e^{\alpha t}\int_t^\infty e^{-\alpha s}p(s)ds.$$

An integration on $[T,t]$ gives

$$\begin{aligned}\log\frac{u(T)}{u(t)} &\le t - T + \alpha\int_T^t e^{\alpha s}\int_s^\infty e^{-\alpha r}p(r)drds\\ &\le t - T + \alpha\int_T^\infty e^{\alpha s}\int_s^\infty e^{-\alpha r}p(r)drds\\ &= t - T + \int_T^\infty \big(1 - e^{-\alpha(s-T)}\big)p(s)ds\\ &\le t + O(1) \qquad \text{as } t \to \infty.\end{aligned}$$

Therefore, $u(t) \ge c_1 e^{-t}$ for some constant $c_1 > 0$. Since $u(t)/e^{-t}$ decreases, we find that (3.14) holds.

Secondly, let $0 < \alpha < 1$. As before we get (3.15). Note that,

$$0 \le \alpha e^{\alpha t}\int_t^\infty e^{-\alpha s}p(s)ds \le \alpha e^{\alpha t}\cdot e^{-\alpha t}\int_t^\infty p(s)ds \le \int_T^\infty p(s)ds.$$

Then, (iv) of Lemma 3.2 implies that for some constant $K > 0$ we obtain

$$-u'(t) \le u(t)\Big[1 + K\alpha e^{\alpha t}\int_t^\infty e^{-\alpha s}p(s)ds\Big].$$

So arguing as in the case that $\alpha > 1$, we can get $u(t) \ge c_2 e^{-t}$ for some constant $c_2 > 0$; and hence (3.14) holds. This completes the proof. $\square$

Proof of Proposition 3.10 for (HL_-). By Lemma 3.9 the function $u(t)/e^{-t}$ is eventually monotone. If $u(t)/e^{-t}$ decreases, then (ii) of Lemma 3.8 implies that $u(t)/e^{-t}$ converges to a positive constant as $t \to \infty$; and so (3.14) holds.

Let us consider the case where $u(t)/e^{-t}$ increases. We may suppose that $u' < 0$ and $u(t)/e^{-t}$ increases on $[T,\infty)$. From (HL_-) we have

$$\big[-u'(t)\big]^\alpha = \alpha\int_t^\infty \big(1 - p(s)\big)u(s)^\alpha ds.$$

The monotonicity of $u(t)/e^{-t}$ implies that

$$\big[-u'(t)\big]^\alpha \ge \alpha e^{\alpha t}u(t)^\alpha\int_t^\infty \Big(e^{-\alpha s} - p(s)e^{-\alpha s}\Big)ds,$$

that is

$$\big[-u'(t)\big]^{\alpha} \geq u(t)^{\alpha}\Big[1-\alpha e^{\alpha t}\int_t^{\infty} p(s)e^{-\alpha s}ds\Big].$$

Notice that

$$\begin{aligned}\alpha e^{\alpha t}\int_t^{\infty} e^{-\alpha s}p(s)ds &\leq \alpha e^{\alpha t}\cdot e^{-\alpha t}\int_t^{\infty} p(s)ds\\ &\leq \int_T^{\infty} p(s)ds \leq 1, \qquad t \geq T,\end{aligned}$$

for sufficiently large T. Therefore (i) and (ii) of Lemma 3.2 implies that,

$$-u'(t) \geq u(t)\Big[1-c\alpha e^{\alpha t}\int_t^{\infty} e^{-\alpha s}p(s)ds\Big], \qquad t \geq T, \tag{3.16}$$

where c is a constant given by

$$c = \begin{cases} 1/\alpha & if\ 0<\alpha<1;\\ 1 & if\ \alpha>1.\end{cases}$$

Dividing the both sides of (3.16) by $u(t)$, and integrating the resulting inequality on $[T,t]$, we obtain

$$\begin{aligned}\log\frac{u(T)}{u(t)} &\geq t-T-c\alpha\int_T^{t}\Big(e^{\alpha s}\int_s^{\infty} e^{-\alpha r}p(r)dr\Big)ds\\ &\geq t-T-c\alpha\int_T^{\infty}\Big(e^{\alpha s}\int_s^{\infty} e^{-\alpha r}p(r)dr\Big)ds\\ &= t-T-c\int_T^{\infty}\Big(1-e^{-\alpha(s-T)}\Big)p(s)ds\\ &= t+O(1) \qquad \text{as } t\to\infty.\end{aligned}$$

Therefore, $u(t) \leq c_2 e^{-t}$ for some constant $c_2>0$. Since $u(t)/e^{-t}$ increases, we find that (3.14) holds. This completes the proof. □

As stated before, it is found that Theorem 3.1 is a direct consequence of Propositions 3.6 and 3.10.

4 The case where $b(t)$ is not of constant sign

In the previous section we have considered our Problem introduced in the Introduction under the condition that $b(t)$ is of constant sign. However, we conjecture that such a signum condition on $b(t)$ may be superflous. So in this section let us consider the Problem without signum conditions on $b(t)$.

We assume throughout this section the next conditions for equation (HL):

(B_1) $\lim_{t\to\infty} b(t) = 0$;

(B_2) $\int^\infty |b(t)|dt < \infty$.

Notice that, in the previous section Section 3, any conditions like (B_1) are not assumed, while condition (B_2), which is essentially the same as condition (A_3), is imposed.

As in Section 3, we find that every positive solution u of (HL) is either an increasing solution or a decreasing solution when (B_1) and (B_2) hold. The following is the main result of this section which gives another affirmative answer to our Problem:

Theorem 4.1. ([9]) *Under assumptions (B_1) and (B_2), every nontrivial solution u of (HL) has the asymptotic form*

$$u(t) \sim ce^t \quad or \quad u(t) \sim ce^{-t} \quad as \quad t \to \infty \tag{4.1}$$

for some constant $c \neq 0$.

More precisely, every positive increasing solution u of (HL) has the former asymptotic form of (4.1), and every positive decreasing solution u of (HL) has the latter.

In Section 3, to see the main result Theorem 3.1 the signum condition of the coefficient function $p(t)$ of $(HL_\pm)$ has been essentially employed. Thus in this section it seems that the method developed in the proof of Theorem 3.1 does not work well. We therefore must find out other methods in proving Theorem 4.1. The key tool in our discussion below is asymptotic analysis of solutions of generalized Riccati equations associated with equation (HL).

It is well know [1, 4, 7] that oscillatory properties of half-linear or linear equations can be clarified by the analysis of generalized Riccati equations associated with them. In fact, there have been lots of results concerning oscillatory and/ or nonoscillatory properties of half-linear equations obtained through the analysis of generalized Riccati equations; see [4, 7]. Here in this thesis, we will prove the main result Theorem 4.1 by applying analysis of generalized Riccati equations associated with (HL).

Related results of generalized Riccati equations are also found in [1, 4, 5, 7, 12].

In Section 4.1 we determine the asymptotic form of positive increasing solutions of (HL); while in Section 4.2 we determine that of positive decreasing solutions of (HL). The proof of Theorem 4.1 will be finished immediately by unifying these results.

4.1 Asymptotic forms of positive increasing solutions of (HL)

We consider asymptotic forms of positive increasing solutions u of (HL); that is, those positive solutions u which satisfy $u'(t) \uparrow \infty$ and $u(t) \uparrow \infty$ as $t \to \infty$.

Proposition 4.2. *Every positive increasing solution u of equation* (HL) *has the asymptotic form*

$$u(t) \sim ce^t \quad \text{as } t \to \infty \qquad \text{for some constant } c > 0.$$

The proof of Proposition 4.2 needs several lemmas:

Lemma 4.3. *Let u be a positive solution of* (HL) *satisfying property* (i) *of Lemma 2.3, and put $w = \big(u'/u\big)^{\alpha}$ for sufficiently large t. Then w satisfies the generalized Riccati equation*

$$w' = \alpha\big(1 + b(t)\big) - \alpha w^{\frac{\alpha+1}{\alpha}}. \tag{4.2}$$

This lemma can be proved by a direct computation.

Lemma 4.4. *Let u be a positive solution of* (HL) *satisfying property* (i) *of Lemma 2.3. Then* $\lim_{t\to\infty} u'(t)/u(t) = 1$.

Proof. Put $q(t) = \big(1 + b(t)\big)^{\alpha/(\alpha+1)}$. Then $\lim_{t\to\infty} q(t) = 1$, and the function $w = \big(u'/u\big)^{\alpha}$ satisfies

$$w' = \alpha\Big(q(t)^{\frac{\alpha+1}{\alpha}} - w^{\frac{\alpha+1}{\alpha}}\Big) \tag{4.3}$$

by Lemma 4.3. It is sufficient to show that $\lim_{t\to\infty} w(t) = 1$. We consider the following three exclusive cases separately:

Case (a): $w(t) \geq q(t)$ near $+\infty$;
Case (b): $w(t) \leq q(t)$ near $+\infty$;
Case (c): $w(t) - q(t)$ changes the sign in any neighborhood of $+\infty$.

Let *Case* (a) occur. By (4.3) we have $w'(t) \leq 0$; so $w(t)$ decreases near $+\infty$. Since $w(t) \geq q(t)$ and $\lim_{t\to\infty} q(t) = 1$, there is a limit $\lim_{t\to\infty} w(t) = L \in [1, \infty)$. Let $t \to \infty$ in (4.3). Then we have $\lim_{t\to\infty} w'(t) = \alpha\big(1 - L^{(\alpha+1)/\alpha}\big)$. Since $w(t)$ is bounded, $\lim_{t\to\infty} w'(t)$ must be 0; which means that $L = 1$. So $\lim_{t\to\infty} w(t) = 1$.

Let *Case* (b) occur. We can show that $\lim_{t\to\infty} w(t) = 1$ similarly.

Finally let *Case* (c) occur. Put $\underline{L} = \liminf_{t\to\infty} w(t)$ and $\overline{L} = \limsup_{t\to\infty} w(t)$. Note that $w'(t) > 0$ [resp. $w'(t) < 0$] if and only if $w(t) < q(t)$ [resp. $w(t) > q(t)$]. Therefore $0 < \underline{L} \leq \overline{L} < \infty$.

To prove $\lim_{t\to\infty} w(t) = 1$, that is $\underline{L} = \overline{L} = 1$, we suppose the contrary that this is not the case.

If $\underline{L} = \overline{L}$, then we can show $\underline{L} = \overline{L} = 1$ as before. So we may assume $\underline{L} < \overline{L}$. From (4.3) and the fact that $0 < \underline{L} < \overline{L} < \infty$ we have $\underline{L} \leq 1 \leq \overline{L}$ (and $\underline{L} < \overline{L}$). Consequently there are three possibilities:

Case (c)-(i): $\underline{L} < 1 < \overline{L}$;
Case (c)-(ii): $\underline{L} < 1 = \overline{L}$;
Case (c)-(iii): $\underline{L} = 1 < \overline{L}$.

Let *Case* (c)-(i) hold. Put $\underline{L} = 1 - \delta$ $(0 < \delta < 1)$. Then there is a sequence $\{t_n\}$ satisfying

$$t_1 < t_2 < \cdots < t_n < t_{n+1} < \cdots; \lim_{n\to\infty} t_n = \infty;$$
$$w'(t_n) = 0, \quad \text{and} \quad w(t_n) < 1 - (\delta/2), \quad n \in \mathbb{N}.$$

By putting $t = t_n$ in (4.3), we get,

$$\begin{aligned} 0 = w'(t_n) &= \alpha\Big[q(t_n)^{\frac{\alpha+1}{\alpha}} - w(t_n)^{\frac{\alpha+1}{\alpha}}\Big] \\ &> \alpha\Big[q(t_n)^{\frac{\alpha+1}{\alpha}} - \Big(1 - \frac{\delta}{2}\Big)^{\frac{\alpha+1}{\alpha}}\Big]. \end{aligned}$$

Let $n \to \infty$ in the above inequality. Then we have a contradiction:

$$0 \geq \alpha\Big[1 - \Big(1 - \frac{\delta}{2}\Big)^{\frac{\alpha+1}{\alpha}}\Big].$$

Therefore, *Case* (c)-(i) does not occur. Similarly, we can show that *Case* (c)-(ii) does not occur. Let *Case* (c)-(iii) hold. Put $\overline{L} = 1 + \delta$ ($\delta > 0$). Then there is a sequence $\{t_n\}$ satisfying

$$
\begin{aligned}
&t_1 < t_2 < \cdots < t_n < t_{n+1} < \cdots; \ \lim_{n\to\infty} t_n = \infty; \\
&w'(t_n) = 0, \quad \text{and} \quad w(t_n) > 1 + (\delta/2), \quad n \in \mathbb{N}.
\end{aligned}
$$

As in the previous Cases, we can get a contradiction.

This completes the proof. □

The following simple lemma is a variant of Gronwall's lemma:

Lemma 4.5. *Let $f, g \in C[t_0, \infty)$, and $c \geq 0$ be a constant such that $f(t), g(t) \geq 0$, and*

$$f(t) \leq c + \int_{t_0}^{t} f(s)ds + \int_{t_0}^{t} g(s)ds, \quad t \geq t_0.$$

Then

$$f(t) \leq ce^{t-t_0} + \int_{t_0}^{t} e^{t-s} g(s)ds, \quad t \geq t_0.$$

Proof. Let us put $H(t) = c + \int_{t_0}^{t} f(s)ds + \int_{t_0}^{t} g(s)ds$. Then $f(t) \leq H(t)$ and

$$H'(t) = f(t) + g(t) \leq H(t) + g(t), \qquad t \geq t_0,$$

by the assumption. Therefore,

$$\left(e^{-t}H(t)\right)' \leq e^{-t}g(t), \qquad t \geq t_0,$$

and so an integration on $[t_0, t]$ gives

$$H(t) \leq ce^{t-t_0} + e^{t}\int_{t_0}^{t} e^{-s}g(s)ds, \qquad t \geq t_0.$$

Since $f(t) \leq H(t)$, the desired estimate of $f(t)$ holds. This completes the proof. □

Now we are in a position to prove Proposition 4.2.

Proof of Proposition 4.2. We may suppose that $u(t) > 0$ and $u'(t) > 0$. Let $w = (u'/u)^\alpha$ as in the proof of Lemma 4.4. We know that $\lim_{t\to\infty} w(t) = 1$. Further put $z(t) = w(t) - 1$. Then $\lim_{t\to\infty} z(t) = 0$ and $z(t)$ satisfies the equation

$$z' = \alpha\bigl(1 + b(t)\bigr) - \alpha\bigl(1 + z\bigr)^{(\alpha+1)/\alpha}. \tag{4.4}$$

Since

$$\bigl(1 + x\bigr)^{\frac{\alpha+1}{\alpha}} = 1 + \frac{\alpha+1}{\alpha}x + \varphi(x), \quad |x| < 1 \tag{4.5}$$

for some continuous function φ with $\varphi(x) = O(x^2),\ x \to 0$, equation (4.4) can be rewritten as follows:

$$z' + \beta z = -\alpha\varphi(z) + \alpha b(t), \quad \beta = 1 + \alpha\ (> 0).$$

This is equivalent to

$$\bigl(e^{\beta t} z\bigr)' = -\alpha e^{\beta t}\varphi(z) + \alpha e^{\beta t} b(t). \tag{4.6}$$

Let us estimate $z(t)$. Since $\lim_{t\to\infty} z(t) = 0$ and $\lim_{x\to 0}\varphi(x)/x = 0$, there is a sufficiently large $T > 0$ satisfying

$$\alpha\bigl|\varphi\bigl(z(s)\bigr)\bigr| \le |z(s)| \quad \text{for } s \ge T.$$

An integration of both the sides of (4.6) on $[T, t]$ gives

$$e^{\beta t} z(t) = c_1 - \alpha\int_T^t e^{\beta s}\varphi(z(s))ds + \alpha\int_T^t e^{\beta s} b(s)ds, \tag{4.7}$$

where $c_1 = e^{\beta T} z(T)$. Therefore,

$$e^{\beta t}|z(t)| \le |c_1| + \int_T^t e^{\beta s}|z(s)|ds + \alpha\int_T^t e^{\beta s}|b(s)|ds.$$

By Lemma 4.5 we have for $t \ge T$,

$$e^{\beta t}|z(t)| \le |c_1| e^{t-T} + \alpha\int_T^t e^{t-s}\cdot e^{\beta s}|b(s)|ds,$$

that is,

$$|z(t)| \leq c_2 e^{-\alpha t} + \alpha e^{-\alpha t} \int_T^t e^{\alpha s} |b(s)| ds \tag{4.8}$$

with some constant $c_2 > 0$.

Recall that,

$$\frac{u'(t)}{u(t)} = w(t)^{1/\alpha} = \big(1 + z(t)\big)^{1/\alpha}. \tag{4.9}$$

Since

$$(1+x)^{1/\alpha} = 1 + \frac{1}{\alpha} x + \rho(x), \quad |x| < 1, \tag{4.10}$$

for some continuous function ρ satisfying $\lim_{x \to 0} \rho(x)/x = 0$, we obtain from (4.9)

$$\begin{aligned} \int_T^t \frac{u'(s)}{u(s)} ds &= \int_T^t \big(1 + z(s)\big)^{1/\alpha} ds \\ &= \int_T^t \Big[1 + \frac{1}{\alpha} z(s) + \rho\big(z(s)\big)\Big] ds, \end{aligned}$$

and so

$$e^{-t} u(t) = u(T) \exp\Big(-T + \frac{1}{\alpha} \int_T^t z(s) ds + \int_T^t \rho(z(s)) ds \Big).$$

To see $u(t) \sim ce^t$ for some constant $c > 0$, it is sufficient to show that $\int^\infty |z(s)| ds < \infty$ and $\int^\infty |\rho(z(s))| ds < \infty$. In the following we will show these facts. By (4.8) and (B_2) we find that,

$$\begin{aligned} \int_T^\infty |z(t)| dt &\leq c_2 \int_T^\infty e^{-\alpha t} dt + \alpha \int_T^\infty e^{-\alpha t} \int_T^t e^{\alpha s} |b(s)| ds dt \\ &\leq \text{const} + \int_T^\infty |b(t)| dt < \infty. \end{aligned}$$

Since we may assume that T is sufficiently large, we find from the property of ρ that,

$$\big|\rho\big(z(t)\big)\big| \leq |z(t)| \quad \text{for } t \geq T.$$

Therefore,

$$\int_T^\infty \big|\rho(z(t))\big| dt \leq \int_T^\infty |z(t)| dt < \infty.$$

This completes the proof of Proposition 4.2. □

4.2 Asymptotic forms of positive decreasing solutions of (HL)

Finally we study asymptotic forms of positive decreasing solutions u of (HL), that is, those positive solutions u which satisfy $u'(t) \uparrow 0$ and $u(t) \downarrow 0$ as $t \to \infty$.

Proposition 4.6. *Every positive decreasing solution u of (HL) has the asymptotic form*

$$u(t) \sim ce^{-t} \quad \text{as } t \to \infty, \qquad \text{for some constant } c > 0.$$

The proof of Proposition 4.6 is carried out by the arguments parallel to that employed in Section 4.1. We must prepare several lemmas.

Lemma 4.7. *Let u be a positive solution of (HL) satisfying property* (ii) *of Lemma 2.3, and put $w = (-u'/u)^{\alpha}$ for sufficiently large t. Then w satisfies the generalized Riccati equation*

$$w' = \alpha w^{(\alpha+1)/\alpha} - \alpha\big(1 + b(t)\big).$$

Lemma 4.8. *Let u be a positive solution of (HL) satisfying property* (ii) *of Lemma 2.3. Then $\lim_{t\to\infty}\big[-u'(t)/u(t)\big] = 1$.*

Proof. Put $q(t) = \big(1 + b(t)\big)^{\alpha/(\alpha+1)}$. Then $\lim_{t\to\infty} q(t) = 1$, and the function $w = (-u'/u)^{\alpha}$ satisfies

$$w' = \alpha\Big(w^{(\alpha+1)/\alpha} - q(t)^{(\alpha+1)/\alpha}\Big) \tag{4.11}$$

by Lemma 4.7. It suffices to show that $\lim_{t\to\infty} w(t) = 1$. We consider the following three exclusive cases separately:

Case (a): $w(t) \geq q(t)$ near $+\infty$;
Case (b): $w(t) \leq q(t)$ near $+\infty$;
Case (c): $w(t) - q(t)$ changes the sign in any neighborhood of $+\infty$.

Let *Case* (a) occur. Since $w'(t) \geq 0$ by (4.11), there is a limit $\lim_{t\to\infty} w(t) \equiv L \in [1, \infty]$. Suppose that $L = +\infty$. Since $(\alpha+1)/\alpha > 1$ and $\lim_{t\to\infty} q(t) = 1$, we find from (4.11) that there is a sufficiently large T satisfying

$$w'(t) \geq \frac{\alpha}{2} w(t)^{\lambda} > 0, \quad t \geq T, \quad \lambda = (\alpha+1)/\alpha > 1.$$

So, $w'(t)w(t)^{-\lambda} \geq \alpha/2$, that is

$$\Big(\frac{w(t)^{1-\lambda}}{1-\lambda}\Big)' \geq \frac{\alpha}{2}, \quad t \geq T.$$

Integrating on $[T, t]$, we obtain

$$\frac{w(T)^{1-\lambda}}{\lambda - 1} \geq \frac{\alpha}{2}(t - T), \quad t \geq T.$$

This is an obvious contradiction. Thus $L \in [1, \infty)$. Letting $t \to \infty$ in (4.11), we get $\lim_{t\to\infty} w'(t) = \alpha\big(L^{\frac{\alpha+1}{\alpha}} - 1\big)$. Then as pointed out before, we have $L = 1$ as desired.

Case (b) can be treated similarly; and so we find that $\lim_{t\to\infty} w(t) = 1$.

Finally let *Case* (c) occur. Put $\underline{L} = \liminf_{t\to\infty} w(t)$ and $\overline{L} = \limsup_{t\to\infty} w(t)$. Note that $w'(t) > 0$ [resp. $w'(t) < 0$] if and only if $w(t) > q(t)$ [resp. $w(t) < q(t)$].

To prove $\lim_{t\to\infty} w(t) = 1$, we suppose the contrary that this is not the case.

If $\underline{L} = \overline{L} \in [0, \infty)$, then we can show $\underline{L} = \overline{L} = 1$ easily. So we may assume that $\underline{L} < \overline{L}$. We find from (4.11) that $0 \leq \underline{L} \leq 1 \leq \overline{L} \leq +\infty$ (and $\underline{L} < \overline{L}$). There are three possibilities:

Case (c)-(i): $0 \leq \underline{L} < 1 < \overline{L} \leq +\infty$;
Case (c)-(ii): $0 \leq \underline{L} < 1 = \overline{L}$;
Case (c)-(iii): $\underline{L} = 1 < \overline{L} \leq +\infty$.

Let *Case* (c)-(i) hold. Put $\underline{L} = 1 - \delta$ $(0 < \delta \leq 1)$. Then, as in the proof of Lemma 4.4, we get a sequence $\{t_n\}$ satisfying

$$t_1 < t_2 < \cdots < t_n < t_{n+1} < \cdots; \quad \lim_{n\to\infty} t_n = \infty;$$
$$w'(t_n) = 0 \quad \text{and} \quad w(t_n) < 1 - \frac{\delta}{2} \quad \text{for } n \in \mathbb{N}.$$

Putting $t = t_n$ in (4.11), and letting $n \to \infty$ in the resulting equation, we get

$$0 \leq \alpha\Big[\Big(1 - \frac{\delta}{2}\Big)^{(\alpha+1)/\alpha} - 1\Big].$$

This is an obvious contradiction. Similarly we can show that *Case* (c)-(ii) does not occur. Let (c)-(iii) hold. Put $\overline{L} = 1 + \delta$ ($\delta > 0$) if $\overline{L} < \infty$. Then as before we can find a sequence $\{t_n\}$ satisfying

$$t_1 < t_2 < \cdots < t_n < t_{n+1} < \cdots ; \quad \lim_{n\to\infty} t_n = \infty;$$
$$w'(t_n) = 0 \quad \text{and} \quad w(t_n) > 1 + \frac{\delta}{2} \quad \text{for } n \in \mathbb{N}.$$

As before we can get a contradiction. The case where $\overline{L} = \infty$ can be treated similarly.
This completes the proof. □

We are now in a position to prove Proposition 4.6.

Proof of Proposition 4.6. We may assume that $u(t) > 0$ and $u'(t) < 0$. Let $w = (-u'/u)^\alpha$ as in the proof of Lemma 4.8, in which we have proved $\lim_{t\to\infty} w(t) = 1$. Put $z(t) = w(t) - 1$. Then $\lim_{t\to\infty} z(t) = 0$, and $z(t)$ satisfies the equation

$$z' = \alpha(1+z)^{(\alpha+1)/\alpha} - \alpha\big(1 + b(t)\big).$$

By (4.5) we can rewrite this equation into

$$z' - \beta z = \alpha\varphi(z) - \alpha b(t), \quad \beta = 1 + \alpha,$$

where $\varphi(x) = O(x^2)$ as $x \to 0$. It follows that

$$\big(e^{-\beta t} z\big)' = \alpha e^{-\beta t}\varphi(z) - \alpha e^{-\beta t} b(t),$$

and an integration on $[t, \infty)$ gives

$$e^{-\beta t} z(t) = -\alpha \int_t^\infty e^{-\beta s}\varphi\big(z(s)\big)ds + \alpha \int_t^\infty e^{-\beta s} b(s)ds. \tag{4.12}$$

As in the proof of Proposition 4.2, for arbitrary number $\varepsilon > 0$ we can find a sufficiently large number $T = T_\varepsilon > 0$ such that

$$\big|\varphi\big(z(s)\big)\big| \le \varepsilon |z(s)|, \quad s \ge T.$$

So, from (4.12) we obtain

$$e^{-\beta t}|z(t)| \le \alpha\varepsilon \int_t^\infty e^{-\beta s}|z(s)|ds + \alpha \int_t^\infty e^{-\beta s}|b(s)|ds, \quad t \ge T. \tag{4.13}$$

Let us denote by $I(t)$ the right-hand side of (4.13). Then

$$e^{-\beta t}|z(t)| \le I(t), \text{ and } I(t) = o(e^{-\beta t}) \text{ as } t \to \infty. \tag{4.14}$$

Since

$$\begin{aligned} -I'(t) &= \alpha\varepsilon e^{-\beta t}|z(t)| + \alpha e^{-\beta t}|b(t)| \\ &\le \alpha\varepsilon I(t) + \alpha e^{-\beta t}|b(t)|, \end{aligned}$$

we obtain

$$\big(e^{\alpha\varepsilon t} I(t)\big)' \ge -\alpha e^{-(\beta-\alpha\varepsilon)t}|b(t)|. \tag{4.15}$$

From now on we fix $\varepsilon > 0$ so small that $\beta - \alpha\varepsilon > 0$. Then by (4.14) $\lim_{t\to\infty} e^{\alpha\varepsilon t} I(t) = 0$. Therefore an integration of (4.15) on $[t, \infty)$ gives

$$0 \le I(t) \le \alpha e^{-\alpha\varepsilon t} \int_t^\infty e^{-(\beta-\alpha\varepsilon)s}|b(s)|ds.$$

By this estimate and the first inequality of (4.14) we find that

$$|z(t)| \le \alpha e^{(\beta-\alpha\varepsilon)t} \int_t^\infty e^{-(\beta-\alpha\varepsilon)s}|b(s)|ds. \tag{4.16}$$

Since $-u'(t)/u(t) = \big(1 + z(t)\big)^{1/\alpha}$, by (4.9) we have

$$-\frac{u'(t)}{u(t)} = 1 + \frac{1}{\alpha} z(t) + \rho\big(z(t)\big), \tag{4.17}$$

where $\lim_{x\to 0} \rho(x)/x = 0$. Integrating (4.17) on $[T, t]$, we obtain

$$\log \frac{u(T)}{u(t)} = t - T + \frac{1}{\alpha} \int_T^t z(s)ds + \int_T^t \rho\big(z(s)\big)ds. \tag{4.18}$$

By (4.16) we find that

$$\begin{aligned} \int_T^\infty |z(s)|ds &\le \alpha \int_T^\infty e^{(\beta-\alpha\varepsilon)s}\Big(\int_s^\infty e^{-(\beta-\alpha\varepsilon)r}|b(r)|dr\Big)ds \\ &\le c_1 \int_T^\infty |b(s)|ds < \infty \end{aligned}$$

for some constant $c_1 > 0$. Similarly, since $\rho(x) = o(x)$ as $x \to 0$, we find that

$$\int_T^\infty |\rho\big(z(s)\big)| ds \le c_2 \int_T^\infty |z(s)| ds < \infty$$

for some constant $c_2 > 0$. Therefore (4.18) implies that

$$\log \frac{u(T)}{u(t)} = t + c_3 + o(1) \text{ as } t \to \infty$$

for some constant $c_3 \in \mathbb{R}$, and so

$$u(t) \sim ce^{-t} \quad \text{as } t \to \infty,$$

for some constant $c > 0$.

This completes the proof of Proposition 4.6. □

As stated before, it is found that Theorem 4.1 is a direct consequence of Propositions 4.2 and 4.6.

Appendix: Generalized hyperbolic functions

As stated in the Introduction, we give a short survey of generalized hyperbolic functions, and the relation between them and solutions of the half-linear equation

$$(|u'|^{\alpha-1}u')' = \alpha|u|^{\alpha-1}u, \quad \alpha > 0, \tag{A.1}$$

which is one of the prototypes of (HL), and is identical with equation (HL_0) in the Introduction.

Definition A.1. *Let* $\alpha > 0$.
(i) *The unique solution* $E(t) \equiv E_\alpha(t)$ *of the initial value problem*

$$\begin{cases} \Big(|E'|^{\alpha-1}E'\Big)' = \alpha|E|^{\alpha-1}E, \quad t \in \mathbb{R}, \\ E(0) = 0, E'(0) = 1 \end{cases} \tag{A.2}$$

is called the generalized hyperbolic sine function (with exponent α*).*
(ii) *The unique solution* $F(t) \equiv F_\alpha(t)$ *of the initial value problem*

$$\begin{cases} \Big(|F'|^{\alpha-1}F'\Big)' = \alpha|F|^{\alpha-1}F, \quad t \in \mathbb{R}, \\ F(0) = 1, F'(0) = 0 \end{cases} \tag{A.3}$$

is called the generalized hyperbolic cosine function (with exponent α*).*

When $\alpha = 1$, obviously $E_\alpha(t)$ and $F_\alpha(t)$ reduce to the hyperbolic sine function $E_1(t) = \sinh t = (e^t - e^{-t})/2$ and the hyperbolic cosine function $F_1(t) = \cosh t = (e^t + e^{-t})/2$, respectively.

The fundamental properties of $E_\alpha = E$ and $F_\alpha = F$, $\alpha > 0$, are as follows:

Proposition A.2. (a) $E'(t) > 0$, $t \in \mathbb{R}$, and $\lim_{t\to\pm\infty} E(t) = \pm\infty$;
(b) $E(-t) = -E(t)$, $t \in \mathbb{R}$;
(c) $\big(E'(t)\big)^{\alpha+1} = |E(t)|^{\alpha+1} + 1$, $t \in \mathbb{R}$;

$$(d) \int_0^{E(t)} \frac{dy}{(1+|y|^{\alpha+1})^{1/(\alpha+1)}} = t, \ t \in \mathbb{R}. \tag{A.4}$$

Proposition A.3. (a) $F'(t) > 0,\ t > 0$, and $\lim_{t\to\pm\infty} F(t) = \infty$;
(b) $F(-t) = F(t), \quad t \in \mathbb{R}$;
(c) $|F'(t)|^{\alpha+1} = F(t)^{\alpha+1} - 1, \quad t \in \mathbb{R}$;

(d) $$\int_1^{F(t)} \frac{dy}{(y^{\alpha+1}-1)^{1/(\alpha+1)}} = |t|,\ t \in \mathbb{R}.$$

In the sequel for simplicity we sometimes denote for $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}$

$$|x|^{\lambda-1}x = x^{\lambda *}.$$

Then the following simple properties hold:

$$(xy)^{\lambda *} = x^{\lambda *}y^{\lambda *} \quad \text{for} \quad x, y \in \mathbb{R};$$
$$x^{\lambda *} > y \quad \text{if and only if} \quad x > y^{1/\lambda *} \quad \text{for } x, y \in \mathbb{R};$$
$$\frac{d}{dx}(x^{\lambda *}) = \lambda |x|^{\lambda-1}; \quad \frac{d}{dx}(|x|^{\lambda}) = \lambda x^{\lambda-1 *}.$$

For example, equation (A.1) can be rewritten as

$$((u')^{\alpha *})' = \alpha u^{\alpha *}.$$

Proof of Proposition A. 2. Let $x(t) \equiv -E(-t)$. Then $x(t)$ satisfies

$$\left(|x'|^{\alpha-1}x'\right)' = \alpha |x|^{\alpha-1}x,$$
$$x(0) = 0,\ x'(0) = 1.$$

Since every initial value problem of (A.1) has unique solution, $x(t) \equiv E(t)$, namely $E(-t) = -E(t)$. The property (b) has been proved.

We multiple the both sides of (A.2) by $E'(t)$, and integrate on $[0, t]$:

$$\int_0^t \left((E'(s))^{\alpha *}\right)' E'(s)ds = \alpha \int_0^t (E(s))^{\alpha *} E'(s)ds.$$

By putting $w(s) = (E'(s))^{\alpha *}$, that is $E'(s) = w(s)^{1/\alpha *}$, we obtain

$$\int_0^t w(s)^{\frac{1}{\alpha} *} w'(s)ds = \alpha \int_0^t (E(s))^{\alpha *} E'(s)ds.$$

Since the left hand-side is

$$\frac{\alpha}{\alpha+1}\Big(|w(t)|^{\frac{\alpha+1}{\alpha}}-1\Big)=\frac{\alpha}{\alpha+1}\Big(|E'(t)|^{\alpha+1}-1\Big),$$

we have

$$|E'(t)|^{\alpha+1}=1+|E(t)|^{\alpha+1},\quad t\in\mathbb{R}.$$

This identity shows that $E'(t)\neq 0$ for any $t\in\mathbb{R}$. Noting that $E'(0)=1$, we find that $E'(t)>0$ by the intermediate value theorem, and the property (c) holds.

The property (c) implies that $E'(t)\geq 1,\ t\in\mathbb{R}$, which shows that $\lim_{t\to\infty}E(t)=\infty$. Since $E(t)$ is an odd function, $\lim_{t\to-\infty}E(t)=-\infty$.

The property (c) implies that

$$E'(t)\big(|E(t)|^{\alpha+1}+1\big)^{-1/(\alpha+1)}=1,\quad t\in\mathbb{R}.$$

So an integration on $[0,t]$ gives the property (d). This completes the proof. □

Proof of Proposition A.3. As in the proof of Proposition A. 2, we can prove $F(-t)=F(t),\ t\in\mathbb{R}$.

Similarly we can find from (A.3) that

$$|F'(t)|^{\alpha+1}=|F(t)|^{\alpha+1}-1,\ t\in\mathbb{R},\tag{A.5}$$

which implies $|F(t)|^{\alpha+1}\geq 1,\ t\in\mathbb{R}$. Noting that $F(0)=1$, we find that $F(t)>0$ and $F(t)\geq 1,\ t\in\mathbb{R}$. So we get the property (c) from (A.5).

Since $F(t)\geq 1,\ t\in\mathbb{R}$, (A.3) implies that

$$\big((F'(t))^{\alpha*}\big)'\geq\alpha,\ t\in\mathbb{R}.$$

An integration on $[0,t],\ t>0$ gives

$$(F'(t))^{\alpha*}\geq\alpha t;\ \text{i.e.,}\ F'(t)\geq\alpha^{1/\alpha}t^{1/\alpha}.$$

Therefore $F'(t)>0,\ t>0$, and $\lim_{t\to\infty}F(t)=\infty$.

The property (d) can be proved as in the proof of Proposition A. 2. This completes the proof. □

The assertion of the following proposition is essentially that of the first half of Fact 1.2 in the Introduction:

Proposition A.4. *Let $\alpha > 0$, and $u_0, u_1 \in \mathbb{R}$. Then the solution u of the initial value problem*

$$\left(|u'|^{\alpha-1}u'\right)' = \alpha|u|^{\alpha-1}u, \tag{A.6}$$
$$u(0) = u_0, \quad u'(0) = u_1 \tag{A.7}$$

has one of the following forms:

(*a*) $u(t) = Ke^t$;
(*b*) $u(t) = Ke^{-t}$;
(*c*) $u(t) = KE(t + t_0)$;
(*d*) $u(t) = KF(t + t_0)$,

where $K, t_0 \in \mathbb{R}$ are some constants, and E and F are generalized hyperbolic sine function $E = E_\alpha$ and the generalized hyperbolic cosine function $F = F_\alpha$, respectively.

Proof. We may assume that $|u_0| + |u_1| > 0$, because the solution u of (A.6)-(A.7) with $u_0 = u_1 = 0$ is $u \equiv 0$ by Lemma 2.1.

As before we find that the solution u of (A.6)-(A.7) satisfies

$$|u'(t)|^{\alpha+1} - |u(t)|^{\alpha+1} = |u_1|^{\alpha+1} - |u_0|^{\alpha+1} \equiv^{put} C. \tag{A.8}$$

The proof is divided into three cases according to the sign of C.

Case 1. The case where $C = 0$. In this case, from (A.8) we obtain

$$|u'(t)| = |u(t)|; \quad \text{and}$$
$$u_1 = \pm\, u_0.$$

If $u_1 = u_0$, then $u(t) = u_0e^t$, and if $u_1 = -u_0$, then $u(t) = u_0e^{-t}$.

Case 2. The case where $C > 0$.

(i) Further suppose $u_0u_1 > 0$. Let $t_0 > 0$ be the number such that

$$(\text{sgn } u_0)\ C^{\frac{1}{\alpha+1}}\ E(t_0) = u_0.$$

We will show that

$$u(t) = (\text{sgn } u_0)\ C^{\frac{1}{\alpha+1}}\ E(t + t_0).$$

To this end, put $y(t) = (\text{sgn } u_0)\, C^{\frac{1}{\alpha+1}}\, E(t+t_0)$. Then obviously y is a solution of (A.6), and $y(0) = u_0$ by the definition of t_0. Employing the properties of E (Proposition A. 2) and (A.8), we can examine $y'(0)$:

$$\begin{aligned}(y'(0)\ (\text{sgn } u_0))^{\alpha+1} &= CE'(t_0)^{\alpha+1} = C(E(t_0)^{\alpha+1} + 1)\\ &= (C^{\frac{1}{\alpha+1}} E(t_0))^{\alpha+1} + C = ((\text{sgn } u_0)\ u_0)^{\alpha+1} + C\\ &= |u_0|^{\alpha+1} + \left(|u_1|^{\alpha+1} - |u_0|^{\alpha+1}\right) = |u_1|^{\alpha+1}.\end{aligned}$$

Therefore $y'(0)\ (\text{sgn } u_0) = |u_1|$, and so

$$y'(0) = |u_1|\ (\text{sgn } u_0) = |u_1|\ (\text{sgn } u_1) = u_1.$$

By the uniqueness of solutions of initial value problems, we find that

$$u(t) \equiv y(t) = (\text{sgn } u_0)\ C^{\frac{1}{\alpha+1}}\ E(t+t_0)$$

as desired.
(ii) Suppose $u_0 u_1 < 0$. Let $t_0 < 0$ be the number such that

$$(\text{sgn } u_1)\ C^{\frac{1}{\alpha+1}}\ E(t_0) = u_0.$$

Then, as in (i) we can show that

$$u(t) = (\text{sgn } u_1)\ C^{\frac{1}{\alpha+1}}\ E(t+t_0).$$

(iii) Suppose $u_0 = 0$ (and $u_1 \neq 0$). Then $u(t) = u_1 E(t)$.
Note that the case $u_1 = 0$ does not occur.
Case 3. The case where $C < 0$. For simplicity we put $C = -C_0$ ($C_0 > 0$). So (A.8) can be rewritten as

$$|u'(t)|^{\alpha+1} - |u(t)|^{\alpha+1} = |u_1|^{\alpha+1} - |u_0|^{\alpha+1} = -C_0.$$

(i) Suppose further $u_0 u_1 > 0$. Let $t_0 > 0$ be the number such that

$$(\text{sgn } u_0)\ C_0^{\frac{1}{\alpha+1}}\ F'(t_0) = u_1.$$

We will show that

$$u(t) = (\text{sgn } u_0)\ C_0^{\frac{1}{\alpha+1}} F(t+t_0). \tag{A.9}$$

Put $y(t) = (\text{sgn } u_0)\, C^{\frac{1}{\alpha+1}} F(t + t_0)$. Then y is a solution of (A.6) and $y'(0) = u_1$. Further we find from the property of F and (A.8)

$$\begin{aligned}\big(y(0)(\text{sgn } u_0)\big)^{\alpha+1} &= C_0 F(t_0)^{\alpha+1} = C_0(F'(t_0)^{\alpha+1} + 1) \\ &= (C_0^{\frac{1}{\alpha+1}} F'(t_0))^{\alpha+1} + C_0 \\ &= ((\text{sgn } u_0)u_1)^{\alpha+1} + (-|u_1|^{\alpha+1} + |u_0|^{\alpha+1}) = |u_0|^{\alpha+1}.\end{aligned}$$

Therefore $y(0)$ $(\text{sgn } u_0) = |u_0| = u_0$ $(\text{sgn } u_0)$; and so $y(0) = u_0$. We have (A.9) as before.
(ii) Suppose $u_0 u_1 < 0$. Then there is a $t_0 < 0$ such that

$$(\text{sgn } u_0)\, C_0^{\frac{1}{\alpha+1}} F'(t_0) = u_1.$$

We can show, as before, that

$$u(t) = (\text{sgn } u_0)\, C_0^{\frac{1}{\alpha+1}} F(t + t_0).$$

(iii) Suppose $u_1 = 0$ (and $u_0 \neq 0$). Then $u(t) = u_0 F(t)$.
Note that the case $u_0 = 0$ does not occur.
This completes the proof of Proposition A. 4. □

The following proposition shows that both $E_\alpha(t)$ and $F_\alpha(t)$, $\alpha > 0$, behave like $ce^t, c = const > 0$, as $t \to \infty$:

Proposition A.5. *Let $\alpha > 0$. For $E(t) = E_\alpha(t)$ and $F(t) = F_\alpha(t)$ we have*

(i) $\displaystyle\lim_{t\to\infty} \frac{E(t)}{e^t} = const \in (0, \infty);$

(ii) $\displaystyle\lim_{t\to\infty} \frac{F(t)}{e^t} = const \in (0, \infty).$

Proof. We may show that

$$\lim_{t\to\infty} (t - \log E(t)) = const \in \mathbb{R}; \text{ and}$$
$$\lim_{t\to\infty} (t - \log F(t)) = const \in \mathbb{R}.$$

(i) By (A.4), for large $t > 0$ satisfying $E(t) > 1$ we have

$$
\begin{aligned}
t - \log E(t) &= t - \int_1^{E(t)} \frac{ds}{s} \\
&= \int_0^{E(t)} \frac{ds}{(s^{\alpha+1}+1)^{1/(\alpha+1)}} - \int_1^{E(t)} \frac{ds}{s} \\
&= \int_0^{1} \frac{ds}{(s^{\alpha+1}+1)^{1/(\alpha+1)}} - \int_1^{E(t)} \Big[\frac{1}{s} - \frac{1}{(s^{\alpha+1}+1)^{1/(\alpha+1)}}\Big] ds.
\end{aligned}
$$

Noting that $E(\infty) = \infty$, it suffices to show that

$$
\int_1^{\infty} \Big[\frac{1}{s} - \frac{1}{(s^{\alpha+1}+1)^{1/(\alpha+1)}}\Big] ds < \infty. \tag{A.10}
$$

By employing the Mean Value Theorem to the function $y \mapsto y^{1/(\alpha+1)}$, we find for $s \geq 1$ that

$$
\begin{aligned}
0 \leq \frac{1}{s} - \frac{1}{(s^{\alpha+1}+1)^{1/(\alpha+1)}} &= \frac{(s^{\alpha+1}+1)^{1/(\alpha+1)} - (s^{\alpha+1})^{1/(\alpha+1)}}{s\ (s^{\alpha+1}+1)^{1/(\alpha+1)}} \\
\leq \frac{1}{\alpha+1} . \frac{1}{s^{\alpha+1}(s^{\alpha+1}+1)^{1/(\alpha+1)}} &\leq \frac{1}{\alpha+1} . \frac{1}{s^{\alpha+2}}.
\end{aligned}
$$

Therefore (A.10) holds.

Similarly we can show that (ii) holds. This completes the proof. □

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